Courant Institute
of Mathematical Sciences

MAGNETO-FLUID DYNAMICS DIVISION

MF-38 NYO-10436

MODERATELY STRONG OBLIQUE SHOCKS AND SIMPLE WAVES IN STEADY MAGNETOHYDRODYNAMIC FLOW

Y. M. Lynn

14 February 1964

AEC Research and Development Report

NEW YORK UNIVERSITY

UNCLASSIFIED

Magneto-Fluid Dynamics Division
Courant Institute of Mathematical Sciences
New York University

Physics and
Mathematics

MF-38 NYO-10436

MODERATELY STRONG OBLIQUE SHOCKS AND SIMPLE WAVES IN STEADY MAGNETOHYDRODYNAMIC FLOW

Y. M. Lynn

14 February 1964

AEC Research and Development Report

Contract No. AT(30-1)-1480

UNCLASSIFIED

Abstract

Steady two dimensional magnetohydrodynamic flow for the general case of nonaligned flow and magnetic field directions, as well as of arbitrary strength of magnetohydrodynamic interactions is considered. An explicit solution describing the physical state across either a family of simple waves or an oblique shock of moderate strength is obtained. It is derived by expansion of the jump conditions of the state variables across an oblique magnetohydrodyanmic shock up to second order with respect to the shock strength which is measured by θ, the deflection angle of the flow direction across the wave. The solution corresponds to a correction of linear characteristics solution for small disturbances. It is shown, to the present approximation, that the position of an oblique magnetohydrodyanmic shock bisects the angle made by upstream and downstream characteristics. Simple solutions for different limiting cases are given. A singularity of the expansion is exhibited and investigated in detail.

Table of Contents

Moderately Strong Oblique Shocks and Simple Waves in Steady Magnetohydrodyanamic Flow

I. Introduction

In this paper we are concerned with a study of moderately strong oblique shocks and simple waves in steady two dimensional flow of an inviscid, compressible, infinitely conducting, polytropic gas in an arbitrarily oriented uniform magnetic field. Only the restricted two dimensional case is considered so that both velocity and magnetic field vectors always lie on one plane, i.e. the (x.y) plane, and there is no change of physical state in the direction normal to the plane. The basic differential equations governing the flow field (i.e. the Lunquist equations) belong to a fully hyperbolic system when the local flow velocity is either "super-fast" (the terminal of the vector $-\vec{U}$ being in region I outside of the characteristics locus[1,2] as shown in Fig. 1) or "sub-slow" (in region III-a or III-b) while they belong to a hyperliptic system when the local flow speed is "sub-fast" and "super-slow" (in region II)[3,4]. In any case, the entire or a part of the flow regime is of hyperbolic character which can be solved by means of shocks and simple waves analogous to steady supersonic gasdynamic flow.

---

1. K.O. Friedrichs, "Nonlinear Wave Motion in Magneto-hydrodynamics", Los Alamos Report, LAMS-2105 (1954). See also K.O. Friedrichs and H. Kranzer, "Nonlinear Wave Motion", NYO-6486-VIII, C.I.M.S., New York University (1958).
2. Y.M. Lynn, Phys. Fluids 5, 626 (1962).
3. H. Grad, Rev. Mod. Phys. 32, 830 (1960).
4. W.R. Sears, Rev. Mod. Phys. 32, 701 (1960).

The magnetohydrodynamic shock jump conditions[*,5-7] are well known. For the solution of isentropic magnetohydrodynamic simple waves describing both rarefaction waves and compression waves before the development of shocks and corresponding to a generalization of the Prandtl-Meyer function in gasdynamics, one has to solve a system of nonlinear differential equations. Because of the inherent difficulties associated with this coupled nonlinear system which contains several basic physical parameters, no explicit analytical solution for simple waves valid over all ranges of relevant parameters is available. One can, at best, reduce[8] the restricted two dimensional problem to first solving a primary system consisting of two coupled nonlinear ordinary differential equations for which no exact analytical solution is possible and numerical methods must be used. Although the solutions for the limiting cases of nearly or pure aligned flow and magnetic field directions, of nearly or purely transverse magnetic field, and of strong or weak magnetohydrodynamic interactions can be given[8-10], they not only are very degenerate and do not represent the underlying physical phenomenon to any generality but also are not sufficient to describe actual physical situations in practice.

---

* See footnote 1, page 4.

5. F. de Hoffmann and E. Teller, Phys. Rev. 80, 692 (1950).
6. L. Helfer, Astrophys. Jour. 117, 177 (1953).
7. R. Lüst, Zeit. für Naturfor. 8a, 277 (1953).
8. Y.M. Lynn, Bull. Am. Phys. Soc. Ser. II, 7, 456 (1962).
9. G.S. Golitsyn, Soviet Phys. -JETP 7, 473 (1958).
10. G.A. Lyubimov, Soviet Phys. -Doklady, 4, 529 (1959).

In view of these facts, this work is motivated by the desire of acquiring a best possible analytical solution for magnetohydrodynamic simple waves of moderate strength, but otherwise covering entire ranges of all other basic physical parameters. The result is obtained simply by expanding the algebraic relations of oblique magnetohydrodynamic shock jump conditions up to second order with respect to the shock strength which is measured by θ, the deflection angle of the flow direction across the wave. The equivalence of these solutions is based on a theorem proved by Lax[11] after a theoretical analysis of a general system of conservation laws to which the Lunquist equations belong that the solutions of shocks and simple waves differ in form only in the third order of their wave strength. Consequently the second order solution of moderately strong shocks for $\theta > 0$ also describes the moderately strong rarefaction simple waves for $\theta < 0$ (see Fig. 2). In this way we get, without really solving the nonlinear system of differential equations, an explicit second order approximate solution for simple waves. It corresponds to a correction of the linear characteristics solution for small disturbances. The solution has also the advantage of being much simpler than the exact solution of an oblique magnetohydrodynamic shock written explicitly in terms of its shock strength. Assuming a single form for both shocks and simple waves, our result can be

---

11. P.D. Lax, Comm. Pure Appl. Math. 10, 537 (1957).

effectively used to construct solutions in good approximation for a wide class of concrete flow problems in steady magnetohydrodynamics, e.g. flow past non-conducting slender objects, shock and simple wave interactions, solar wind flow in astrophysical and geophysical problems etc.

In section II we start with a brief discussion of characteristics and then formulate the jump conditions across a general oblique magnetohydrodynamic shock of finite strength from which the first and second order expansion coefficients of various state variables are determined. It is shown explicitly that the second order solution of the shock position bisects the upstream characteristics and the first order downstream characteristics. Solutions of simpler form for the special cases of gasdynamic limit, of aligned field flow and of switch on waves are obtained in section III by taking various limits of the general solution. Section IV discusses in detail the singularity of the expansion at which the general approximate solution fails to exist. It renders the straightforward expansion procecure to be no longer appropriate and a refined approach is necessary. Several general remarks are given in section V. We furnish in the appendix some intermediate calculations in the derivation of the expansion coefficients.

II. General Solution

Before carrying out the formal expansions of jump conditions of the physical state across a stationary magnetohydrodynamic shock, we first discuss briefly in the following several fundamental properties of the characteristics associated with the basic system of governing nonlinear differential equations. These are essential to the understanding of simple waves. By the application of the theory of characteristics to Lunquist's equations written in Cartesian coordinates, we obtain*

$$(udy-vdx)^2[(udy-vdx)^2 - (a^2+b^2)(dx^2+dy^2)] + a^2(dx^2+dy^2)(b_x dy-b_y dx)^2 = 0 \tag{1}$$

where a and b are gasdynamic sound speed, $\sqrt{dP/d\rho}$, and Alfvén speed, $B/\sqrt{\mu\rho}$, respectively, u, v and $b_x = B_x/\sqrt{\mu\rho}$, $b_y = B_y/\sqrt{\mu\rho}$ are x- and y- components of the velocity $\vec{U}$ and the Alfvén velocity $\vec{b} = \vec{B}/\sqrt{\mu\rho}$ respectively. The slope of the characteristics $\zeta = dy/dx$, at any point $P(x,y)$ in the flow field can be easily determined from Eq. (1) which is a biquadratic polynomial of ζ. Let us depict at P the tangents of the characteristics, of the streamline and of the magnetic flux line as shown in Fig. 3 and define the angles (ranged from $-\pi$ to π and measured counterclockwise to be positive as follows:

---

* See footnote 8, page 5.

σ = the angle made by the flow direction (tangent of streamline) from the positive x-axis.

ψ = the angle made by the magnetic field direction from the flow direction.

ω = the angle made by the characteristics direction from the flow direction.

It is clear on physical grounds that ω must be a function of local physical variables alone, independent of the choice of spatial coordinate system. This can be easily achieved by change of variables. We see from Fig. 3

$$\zeta = \frac{dy}{dx} = \tan(\sigma+\omega) \tag{2}$$

$$u = U\cos\sigma, \qquad v = U\sin\sigma$$

$$b_x = b\cos(\sigma+\psi), \quad b_y = b\sin(\sigma+\psi).$$

Let us now introduce

$$M^2 = \frac{U^2}{a^2} = \frac{\rho U^2}{\gamma P}, \qquad A^2 = \frac{U^2}{b^2} = \frac{\mu\rho U^2}{B^2}$$

being the squares of the Mach number and Alfven number respectively. These, together with ψ, are the three dimensionless parameters characterizing the state of fluid.

In terms of the above notations, Eq. (1) becomes

$$M^2A^2\sin^4\omega - (M^2+A^2)\sin^2\omega + \sin^2(\omega-\psi) = 0 \tag{3}$$

from which $\omega = \omega(M,A,\psi)$ is determined and is seen to be independent of σ. Alternatively, one may also obtain Eq.(3) from the normal wave speed equation without referring to any coordinate system as shown by Friedrichs*, i.e.

$$(U_n^2-a^2)(U_n^2-b_n^2) = U_n^2 b_t^2 \tag{4}$$

where n and t are subscripts denoting the normal and tangential components with respect to the wave front. Hence

$$U_n = U \sin \omega$$

$$b_n = b \sin(\omega-\psi), \qquad b_t = b \cos(\omega-\psi).$$

Substituting these into Eq. (4) results in precisely Eq. (3). More explicitly, let

$$\xi = \tan \omega \tag{5}$$

Eq. (3) reads

$$\begin{aligned}(M^2A^2-M^2-A^2+\cos^2\psi)\xi^4 - \sin 2\psi\ \xi^3 + (1-M^2-A^2)\xi^2 \\ - \sin 2\psi\xi + \sin^2\psi = 0.\end{aligned} \tag{6}$$

Hence $\xi = \xi(M,A,\psi)$ is governed by a biquadratic equation which may possess either four real roots or two real and two

---

* See footnote 1, page 4.

complex roots with each of the real roots corresponding to a real characteristics. The equation for the characteristics locus (see Fig. 1) can be obtained by setting the discriminant of Eq. (6) to zero[12]. The characteristics locus, consisting of the outer convex fast wave front and the inner slow wave front (two cusped triangles connected by a straight line at their innermost vertices), is indeed the boundary separating regions in which there exist four real roots (i.e. four real characteristics) from that in which there exist only two real roots (i.e. two real characteristics). Since the hyperbolicity of the basic system of differential equations is determined by the number of real characteristics in existence, we have*

(a) When the terminal of $-\vec{U}$ is in either one of the regions I, IIIa and IIIb as shown in Fig. 1, the flow field belongs to a fully hyperbolic regime. There exist two fast and two slow waves in region I and four slow waves in region IIIa, or IIIb.

(b) When the terminal of $-\vec{U}$ is in region II as shown in Fig. 1, the flow field belongs to a hyperliptic regime. There exist only two slow waves.

In conclusion, there are generally four distinct characteristic angles, ω_i with $i = 1$ to 4, describing the wave inclinations. Furthermore, on each side of the velocity vector, only one of each pair of fast or slow waves may appear.

---

12. c.f. footnote 2, page 4, where a different, but equivalent method was used.

* See footnotes 3 and 4, page 4.

To be specific, the indices are so designated that the relation

$$\omega_1 > \omega_2 \geq 0 \geq \omega_3 > \omega_4 \tag{7}$$

holds always (see Fig. 3). As a consequence, 1 and 2 always denote waves on the positive side of the streamline, while 3 and 4 always denote those on the negative side of the streamline. Moreover, 1 and 4 denote either fast waves (whenever they exist) or slow waves unless the flow regime is hyperliptic for which 1 and 4 must be omitted. 2 and 3 denote slow waves always. In general Eqs. (5 and 2) must be written, for $i = 1$ to 4, as

$$\xi_i = \tan \omega_i \tag{8}$$

being the roots of Eq. (6) and

$$\zeta_i = \left(\frac{dy}{dx}\right)_i = \tan(\sigma + \omega_i) = \frac{v + u\xi_i}{u - v\xi_i} \quad . \tag{9}$$

Note here that, in contrast with the familiar case in gas-dynamics, both

$$\omega_1 > \frac{\pi}{2} \qquad \text{and} \quad \omega_4 < -\frac{\pi}{2}$$

are possible, they correspond to upstream directed fast or slow waves.

More schematically one may write the characteristics equations in Eq. (9) as

$$y_{\alpha_i} = \zeta_i \, x_{\alpha_i} \qquad \text{for} \qquad C_i, \quad i = 1 \text{ to } 4 \tag{10}$$

where α_i denotes the varying parameter along each characteristics, C_i. Eq. (10) are exact equations governing the flow field, each of them is an ordinary differential equation along one of the families of C_i that covers the entire domain of the physical (x,y) plane. Apparently only two of C_i are independent for there are only two independent variables. We are interested here only in the simple waves which, by the general theory, are always adjacent to a constant state. Hence one of C_i describes a family of straight characteristics, these are simple wave surfaces along which the state of fluid remains constant. The cross characteristics are given by one of the other members in Eq. (10).

The jump conditions of the physical state across a stationary magnetohydrodynamic shock of arbitrary strength expressed in terms of the components of $\vec{B}$ and $\vec{U}$ normal and tangential to (denoted by subscripts n and t respectively) the shock surface together with P and ρ, read*

$$[B_n] = 0 \tag{11}$$

$$[U_n B_t - U_t B_n] = 0 \tag{12}$$

$$[\rho U_n^2 + P + \frac{B_t^2 - B_n^2}{2\mu}] = 0 \tag{13}$$

* See, e.g. footnote 1, page 4.

$$[\rho U_n U_t - \frac{B_n B_t}{\mu}] = 0 \tag{14}$$

$$[\rho U_n] = 0 \tag{15}$$

$$[U_n \left\{ \frac{\rho}{2}(U_n^2+U_t^2) + \frac{\gamma}{\gamma-1} P \right\} + \frac{B_t}{\mu} (U_n B_t - U_t B_n)] = 0 \tag{16}$$

where [] denotes the jump of the physical quantity inside it across the shock surface. The equation of state for a polytropic gas that $e = \frac{1P}{\gamma-1\rho}$ has been used to obtain Eq.(16). Let us choose, without loss of generality, the upstream flow direction as a fixed reference line with respect to which all angles are measured (counterclockwise to be positive) and define

β = the shock angle

θ = the deflection angle of the flow direction across the wave.

ψ_I and ψ_F = the angles made by the upstream flow direction with the upstream and downstream magnetic field directions respectively.

Then we have (see Fig. 4), ahead of the shock

$$U_{I,n} = U_I \sin\beta \quad , \quad U_{I,t} = U_I \cos\beta$$

$$B_{I,n} = B_I \sin(\beta-\psi_I) \ , \quad B_{I,t} = B_I \cos(\beta-\psi_I)$$

and behind the shock

$$U_{F,n} = U_F \sin(\beta-\theta) \quad , \qquad U_{F,t} = U_F \cos(\beta-\theta)$$

$$B_{F,n} = B_F \sin(\beta-\psi_F) \ , \qquad B_{F,n} = B_F \cos(\beta-\psi_F).$$

The dimensionless parameters characterizing the uniform state ahead of the shock are $M_I = \frac{U_I}{a_I}$, $A_I = \frac{U_I}{b_I}$ and ψ_I . We introduce the dimensionless variables specifying the ratios of the magnitude of physical variables across the shock

$$\overline{U} = \frac{U_F}{U_I} \quad , \quad \overline{\rho} = \frac{\rho_F}{\rho_I} \ , \quad \overline{P} = \frac{P_F}{P_I} \ , \quad \overline{B} = \frac{B_F}{B_I} \ .$$

In terms of these notations, Eqs. (11 to 16) become

$$\sin(\beta-\psi_I) = \overline{B} \sin(\beta-\psi_F) \tag{17}$$

$$\sin \psi_I = \overline{U}\ \overline{B} \sin(\psi_F-\theta) \tag{18}$$

$$\frac{1}{\gamma M_I^2} + \sin^2\beta + \frac{1}{2A_I^2} \cos 2(\beta-\psi_I) = \frac{\overline{P}}{\gamma M_I^2} + \overline{\rho}\ \overline{U}^2 \sin^2(\beta-\theta) + \frac{\overline{B}^2}{2A_I^2} \cos 2(\beta-\psi_F) \tag{19}$$

$$\sin 2\beta - \frac{1}{A_I^2} \sin 2(\beta-\psi_I) = \overline{\rho}\ \overline{U}^2 \sin 2(\beta-\theta) - \frac{\overline{B}^2}{A_I^2} \sin 2(\beta-\psi_F) \tag{20}$$

$$\sin \beta = \overline{\rho}\ \overline{U} \sin(\beta-\theta) \tag{21}$$

$$\left(1 + \frac{2}{\gamma-1}\frac{1}{M_I^2}\right)\sin\beta + \frac{2}{A_I^2}\sin\psi_I\cos(\beta-\psi_I)$$

$$= \overline{U}\left\{\left(\overline{\rho}\ \overline{U}^2 + \frac{2}{\gamma-1}\frac{\overline{P}}{M_I^2}\right)\sin(\beta-\theta) + \frac{2\overline{B}^2}{A_I^2}\sin(\psi_F-\theta)\cos(\beta-\psi_F)\right\}. \tag{22}^*$$

Since a shock is characterized by one parameter only, it is completely determined if any one of the dimensionless quantities, namely $\overline{\rho}$, $\overline{U}$, $\overline{B}$, $\overline{P}$, ψ_F, β and θ, is specified which may in turn be considered as a measure of the shock strength. We first solve $\overline{U},\overline{\rho},\overline{B}$, and $\overline{P}$ explicitly in terms of the angles β, ψ_F and θ. Eq. (17) gives

$$\overline{B} = \frac{\sin(\beta-\psi_I)}{\sin(\beta-\psi_F)} \tag{23}$$

We obtain from Eq. (21)

---

13. An extension of the present work to the general two-dimensional problem is immediate. Because of no variation in transverse direction (i.e. z-direction) the wave fronts are always perpendicular to (x,y) plane. Then $\overline{U}$ and $\overline{B}$ have in general two tangential components, namely the t_1-component lying on (x,y) plane and the t_2-component in z-direction. Two additional variables, e.g. η and ζ being the polar angles made by $\overline{U}$ and $\overline{B}$ respectively with the z-axis, are necessary to characterize the physical state. Substituting the n-, t_1- and t_2- (or z-) components of $\overline{U}$ and $\overline{B}$ into Eqs. (11 to 16) yields eight equations instead of six in Eqs. (17 to 22). The subsequent expansion calculations are straightforward though considerably more tedious and the solution becomes much more complicated. Note the mode of transverse wave is allowed in this general case. Our solution for the restricted two dimensional problem corresponds to $\eta \equiv \frac{\pi}{2}$ and $\zeta \equiv \frac{\pi}{2}$.

$$\bar{\rho}\ \bar{U} = \frac{\sin\beta}{\sin(\beta-\theta)} \tag{24}$$

Substituting Eqs. (23 and 24) into Eq. (20) yields

$$\bar{U} = \frac{\cos\beta}{\cos(\beta-\theta)} \left[1 + \frac{2}{A_I^2} \frac{\sin(\psi_F-\psi_I)}{\sin 2\beta} \frac{\sin(\beta-\psi_I)}{\sin(\beta-\psi_F)}\right] \tag{25}$$

and consequently from Eq. (24)

$$\bar{\rho} = \frac{\tan\beta}{\tan(\beta-\theta)} \left[1 + \frac{2}{A_I^2} \frac{\sin(\psi_F-\psi_I)}{\sin 2\beta} \frac{\sin(\beta-\psi_I)}{\sin(\beta-\psi_F)}\right]^{-1} \tag{26}$$

Substituting Eqs. (23, 25, and 26) into Eq. (19) yields

$$\begin{aligned} \frac{\bar{P}-1}{\gamma M_I^2} &= \sin^2\beta\left[1 - \frac{\tan(\beta-\theta)}{\tan\beta}\right] \\ &+ \frac{1}{2A_I^2}\left\{1 - \frac{\sin(\beta-\psi_I)}{\sin(\beta-\psi_F)}\left[2\sin(\psi_F-\psi_I)\tan(\beta-\theta) + \frac{\sin(\beta-\psi_I)}{\sin(\beta-\psi_F)}\right]\right\}. \end{aligned} \tag{27}$$

The other two independent equations, consisting of only the angles β,θ and ψ_F, are obtained as follows. Substituting Eqs. (23 and 25) into Eq. (18) yields

$$\frac{\sin\psi_I}{\sin(\psi_F-\theta)}$$

$$= \frac{\sin(\beta-\psi_I)}{\sin(\beta-\psi_F)} \frac{\cos\beta}{\cos(\beta-\theta)} \left[1 + \frac{2}{A_I^2} \frac{\sin(\psi_F-\psi_I)}{\sin 2\beta} \frac{\sin(\beta-\psi_I)}{\sin(\beta-\psi_F)} \right] . \tag{28}$$

Substituting Eqs. (23, 25, 26, and 27) into Eq. (22) would yield immediately the other equation of conservation of energy across the shock. It has a rather lengthy form and can actually be replaced here by a much simpler equation

$$\frac{P_F}{P_I} \left(\frac{\rho_I}{\rho_F}\right)^\gamma = \frac{\bar{P}}{\bar{P}^\gamma} = 1 \tag{29}$$

corresponding to the isentropic condition throught the entire flow. This is permissible because the entropy remains constant in the flow up to second order in wave strength* and we wish to obtain the solution only up to this order of aproximation. This can also be easily realized by the fact that the second order solution for isentropic simple waves takes the same form as that for shocks. Eq. (22) is needed only when we wish to include in the solution the third and (or) higher order terms in wave strength for which the entropy does undergo change across a shock. Obviously the solution thus obtained no longer gives a correct description of simple waves. It may be remarked

* See footnote 11, page 6.

that the above shock relations are given in a form so that the gasdynamic solution corresponding to $A_I \longrightarrow \infty$ can be easily visualized.

To find the first and second order solutions of shock jump conditions, we take θ as the small parameter measuring the wave strength and expand the dimensionless physical variables with respect to it in asymptotic series as

$$\bar{U} = 1 + \sum_{n=1}^{\infty} u_n \theta^n \tag{30}$$

$$\bar{B} = 1 + \sum_{n=1}^{\infty} b_n \theta^n \tag{31}$$

$$\bar{\rho} = 1 + \sum_{n=1}^{\infty} r_n \theta^n \tag{32}$$

$$\bar{P} = 1 + \sum_{n=1}^{\infty} p_n \theta^n \tag{33}$$

$$\psi_F = \psi_I + \sum_{n=1}^{\infty} \psi_n \theta^n \tag{34}$$

$$\beta = \beta_o + \sum_{n=1}^{\infty} \beta_n \theta^n \ . \tag{35}$$

Substituting these expressions into Eqs. (23, 25 to 29) and equating terms on each side of order θ and θ^2 respectively, we arrive at two systems of equations for the first and second order solutions. The coefficients of expansion are then determined explicitly. It is of interest to note that the shock disappears, or the shock strength becomes zero when $\theta = 0$, hence β_o must be determined from the first order system of equations. It is actually the only first order solution describing the shock inclination, since β_1 is cancelled out exactly in the first order system. Therefore the shock position is independent of θ within the first order of approximation. In fact β_o can be identified with the upstream characteristic angle as it will be shown below. Similarly β_1 corresponds to the only second order coefficient to be determined from the second order system of equations in which β_2 does not appear.

(1) First order solution (for derivation see appendix)

Let us denote $t = \tan\beta_o$ and $s = \tan\psi_I$, we find

$$\psi_1 = \frac{(t-s)[M_I^2 t^2(1+s^2)-(t-s)^2]}{t^2(1+s^2)(M_I^2 t+s-t)} \tag{36}$$

$$r_1 = \frac{M_I^2(1+st)}{M_I^2 t+s-t} \tag{37}$$

$$u_1 = \frac{s-t-M_I^2 t^2 s}{t(M_I^2 t+s-t)} \tag{38}$$

$$b_1 = \frac{(1+st)[M_I^2 t^2(1+s^2)-(t-s)^2]}{t^2(1+s^2)(M_I^2 t+s-t)} \tag{39}$$

$$p_1 = \frac{\gamma M_I^2(1+st)}{M_I^2 t+s-t} \tag{40}$$

and $\beta_o = \tan^{-1}t$ is determined from the compatibility equation

$$M_I^2 A_I^2 t^4(1+s^2) - (M_I^2+A_I^2)t^2(1+s^2)(1+t^2) + (1+t^2)(t-s)^2 = 0 \tag{41}$$

which can also be expressed in terms of the angles β_o and ψ_I as

$$M_I^2 A_I^2 \sin^4\beta_o - (M_I^2+A_I^2)\sin^2\beta_o + \sin^2(\beta_o-\psi_I) = 0. \tag{42}$$

A comparison of Eq. (42) with Eq. (3) shows

$$\beta_o = \omega_I \tag{43}$$

hence the position of shock, to the first order approximation, coincides with that of upstream characteristics. Also

$$t = \tan\omega_I = \xi_I \tag{44}$$

(c.f. Eqs. (5 and 6)). The explicit solution of $t = t(M_I, A_I, s)$ obtained from Eq. (41) is very complicated in general. We shall retain t throughout in this paper realizing that it may take, after solving Eq. (41), four distinct values. The root pertains to fast or slow waves in accordance with $M_I^2 t^2/1+t^2 > 1$, or < 1 (i.e. $U_n > a_I$ or $< a_I$). We get from

Eq. (41)

$$A_I^2 = \frac{(1+t^2)[M_I^2 t^2(1+s^2)-(t-s)^2]}{t^2(1+s^2)[M_I^2 t^2-(1+t^2)]} \tag{45}$$

or from Eq. (42)

$$A_I^2 = \frac{M_I^2 \sin^2\omega_I - \sin^2(\omega_I-\psi_I)}{\sin^2\omega_I(M_I^2\sin^2\omega_I - 1)} \quad . \tag{46}$$

In all our solutions Eq. (45) is used to eliminate A_I^2 throughout. This enables us to order terms systematically according to successive powers of M_I^2 which is particularly convenient in the derivation of complicated second order solution.

To express the first order solution in terms of M_I, ω_I and ψ_I explicitly, we have

$$\psi_I = D^{-1}\left[1 - \frac{\sin^2(\omega_I-\psi_I)}{M_I^2\sin^2\omega_I}\right]\tan(\omega_I-\psi_I) \tag{47}$$

$$r_1 = D^{-1} \tag{48}$$

$$u_1 = D^{-1}[\cot\omega_I(\sin^2\omega_I - M_I^{-2})\tan(\omega_I-\psi_I) - \sin^2\omega_I] \tag{49}$$

$$b_1 = D^{-1}\left[1 - \frac{\sin^2(\omega_I-\psi_I)}{M_I^2\sin^2\omega_I}\right] \tag{50}$$

$$p_1 = \gamma D^{-1} \tag{51}$$

where

$$D = (\sin^2\omega_I - M_I^{-2})\tan(\omega_I-\psi_I) + \sin\omega_I\cos\omega_I \ . \tag{52}$$

It can be easily verified from here that the first order solution agrees precisely with the linear solution of characteristics for small disturbances. The latter can also be obtained directly from the basic system of differential equations as shown by Friedrichs*. In particular, the parameter for the wave strength, k, given by Friedrichs can be easily determined, e.g. by comparing his solution of the density jump written in our notations

$$\delta\rho = \frac{k\rho_I^2 U_I^2 M_I^2 \sin^4\omega_I \cos^2(\omega_I-\psi_I)}{M_I^2\sin^2\omega_I - \sin^2(\omega_I-\psi_I)} = \frac{k\rho_I^2 U_I^2 M_I^2 t^4(1+st)^2}{(1+t^2)^2[M_I^2t^2(1+s^2)-(t-s)^2]}$$

with that given in Eq. (37) or (48)

$$\delta\rho = \rho_F - \rho_I = \rho_I r_1 \theta = \rho_I D^{-1}\theta = \frac{\rho_I M_I^2(1+st)\theta}{M_I^2 t+s-t} .$$

Hence

$$k = \frac{[M_I^2\sin^2\omega_I - \sin^2(\omega_I-\psi_I)]\theta}{D\rho_I U_I^2 M_I^2 \sin^4\omega_I \cos^2(\omega_I-\psi_I)} = \frac{(1+t^2)^2[M_I^2t^2(1+s^2)-(t-s)^2]\theta}{\rho_I U_I^2 t^4(1+st)(M_I^2t+s-t)} \tag{53}$$

$k \ll 1$ in general unless $1 + st \cong 0$ or $M_I^2 t + s - t \cong 0$. These conditions are investigated in detail in later sections.

The presentation of the first order solution here is not just for the sake of completeness, it also serves as a reference in terms of which a concise form of the second order solution can be given.

---

* See footnote 1, page 4.

(2) Second order solution (for derivation see appendix)

The explicit expressions of β_1 and ψ_2 written in terms of the known upstream values M_I, s and t are

$$\beta_1 = \frac{(1+st)M_I^2 tG}{4(M_I^2 t+s-t)} \tag{54}$$

and

$$\begin{aligned}
\psi_2 = \frac{s[M_I^2 t^2(1+s^2) - (t-s)^2]}{4t^4(1+s^2)(M_I^2 t+s-t)^3} \Bigg\{ & \Bigg\{ M_I^2 t^2[2(t-s)^2 + (1+t^2)(1+st)] \\
& - 2(1+t^2)(t-s)^2 \Bigg\} M_I^2 tG - \frac{4(t-s)}{1+s^2} \Bigg\{ M_I^4 t^3(1+s^2)(2t-s) \\
& - M_I^2 t(t-s)[(t-s)^2 + 2t^2(1+s^2)] + (t-s)^2(t^2-2st-1) \Bigg\} \Bigg\}
\end{aligned} \tag{55}$$

where

$$G = \frac{3M_I^4 t^4(1+s^2)-M_I^2 t^2[6(t-s)^2+(2-\gamma)(1+st)^2]+3(1+t^2)(t-s)^2}{M_I^4 t^4(1+s^2)+M_I^2 t^2(t-s)(st^2+2s-t)-s(t-s)(1+t^2)^2} \tag{56}$$

r_2, u_2 and b_2, expressed in terms of s and t together with the hitherto known values ψ_1, β_1 and ψ_2, are

$$\begin{aligned}
r_2 = \frac{(1+s^2)t\ \psi_2}{s(t-s)} + \frac{(1+s^2)(1+st)t\ \psi_1^2}{s(t-s)^2} - \left[\frac{1}{t^2} + \frac{(1+s^2)\ \psi_1}{(t-s)^2}\right](1+t^2)\beta_1 \\
+ \frac{s-t}{st^2}
\end{aligned} \tag{57}$$

$$u_2 = \frac{1}{2} - \frac{(1+s^2)t\,\psi_2}{s(t-s)} + \frac{(1+s^2)t\,\psi_1^2 + (s-2t)(1+s^2)\psi_1 + t-s}{s^2(t-s)} + \frac{(1+s^2)(1+t^2)\psi_1\beta_1}{(t-s)^2} \tag{58}$$

and

$$b_2 = \frac{(1+st)\psi_2}{t-s} + \left\{ \left[\frac{1}{2} + \left(\frac{1+st}{t-s}\right)^2\right]\psi_1 - \frac{(1+s^2)(1+t^2)\beta_1}{(t-s)^2} \right\}\psi_1 \tag{59}$$

also

$$p_2 = \gamma\left[r_2 + \frac{\gamma-1}{2}\,\frac{M_I^4(1+st)^2}{(M_I^2 t+s-t)^2}\right] . \tag{60}$$

Certainly we may, by the use of previously given expressions of ψ_1, β_1 and ψ_2 in Eqs. (36, 54, and 55) together with Eq. (56) express r_2, u_2, b_2 and p_2 more explicitly in terms of the basic parameters M_I, s and t exclusively. The final expressions appear to be rather lengthy, they are not necessary and are thereby omitted.

By the use of the above approximation solution, a flow field constituted by simple waves can be constructed as follows. Let us consider, e.g. the centered rarefaction waves $(\theta < 0)$. We first determine the upstream (first) characteristics pertaining to the known upstream state from Eq. (6). For a deflection angle of the flow direction across the entire wave zone, θ, the fluid state at downstream is given by the second order approximate solution. Then we determine the downstream (last) characteristics to which the downstream state variables must be attached. Hence the continuous change of state across a simple wave zone from

its first characteristics is replaced here by only one final state at the last characteristics. Even in the case of strong simple waves with θ being so large that the second order solution is no longer adequate, we may divide θ in segments, i.e. let $\theta = n\Delta\theta$ with $\Delta\theta = \frac{\theta}{n}$ being a small angle for which the approximate solution may be used. Applying the above mentioned procedure successively across each $\Delta\theta$ would yield a sufficiently accurate approximate solution describing the entire simple wave zone. Since simple waves are governed by one parameter only, the inclination of the wave surface (characteristics) alone suffices to characterize each wavelet completely. As a consequence the solution for non-centered rarefaction waves, as well as for compression waves can be constructed in a similar way.

The physical state behind the wave is perturbed about that ahead of it for small θ in the above calculations, therefore we have considered only the case of small departures of the state variables across the wave. This is obvious for the simple waves since the physical state must change continuously from that in upstream through the wave. On the other hand, our solution describes only one of two possible families of oblique magnetohydrodynamic shocks, namely, the weak family that has zero shock strength at $\theta = 0$. Another family of strong shocks at small θ, having finite shock strength hence finite jumps of the state variables even at $\theta = 0$, is also possible. Because of the presence of the magnetic field which may result in a jump

in tangential momentum across the shock surface, the strong magnetohydrodynamic shock at $\theta = 0$ is not generally normal to the upstream (and also the downstream) flow direction, in contrast with the case in gasdynamics. For the reason of their finite wave strength, the strong family of shocks will not be investigated here. There is also no mention about such interesting questions as the shock stability criteria and the maximum allowable flow deflection angle beyond which an attached oblique shock no longer exists. These will be reported in a succeeding paper where a general discussion of oblique magnetohydrodynamic shocks of finite strength, as well as comparison of the approximate first and second order solutions with the exact solution for various shock strengths , will also be presented.

We shall now show that the second order solution of the shock position bisects the angle made by the upstream characteristics and the first order solution of downstream characteristics (see Fig. 5). In other words

$$\beta = \omega_I + \beta_1\theta = \frac{1}{2}\left[\omega_I+(\omega_F+\theta)\right] \tag{61}$$

where ω_F, the angle made by the first order downstream characteristics and the downstream flow direction, is determined by the following equation (c.g. Eq. (3))

$$M_F^2A_F^2\sin^4\omega_F - (M_F^2+A_F^2)\sin^2\omega_F + \sin^2(\omega_F-\psi_F) = 0 \tag{62}$$

with M_F, A_F and ψ_F being the corresponding first order downstream quantities. We define Ω by

$$\Omega\theta = \omega_F - \omega_I \quad . \tag{63}$$

Eq. (61) becomes

$$2\beta_1 = 1 + \Omega \tag{64}$$

and can be proved as follows. By the use of Eqs. (30, 31, 32, 34 and 63), we have

$$M_F^2 = \frac{U_F^2}{a_F^2} = M_I^2 \frac{U_F^2}{U_I^2} \frac{a_I^2}{a_F^2} = M_I^2 \left\{ 1+[2u_1-(\gamma-1)r_1]\theta+O(\theta^2) \right\}$$

$$A_F^2 = \frac{U_F^2}{b_F^2} = A_I^2 \frac{U_F^2}{U_I^2} \frac{b_I^2}{b_F^2} = A_I^2[1+(2u_1-2b_1+r_1)\theta+O(\theta^2)]$$

$$\sin^2\omega_F = \sin^2\omega_I[1+2\Omega \cot \omega_I\cdot\theta+O(\theta^2)]$$

and

$$\sin^2(\omega_F-\psi_F) = \sin^2(\omega_I-\psi_I)[1+2(\Omega-\psi_1)\cot(\omega_I-\psi_I)\cdot\theta+O(\theta^2)].$$

Substituting these expressions into Eq. (62) and subtracting Eq. (42) from it, we get the equation for the first order terms

$$\begin{aligned} &M_I^2A_I^2\sin^4\omega_I[4\Omega \cot \omega_I+ 4u_1 + (2-\gamma)r_1 - 2b_1] \\ &- \sin^2\omega_I \left\{ 2(M_I^2+A_I^2)\Omega \cot \omega_I+ M_I^2[2u_1-(\gamma-1)r_1] + A_I^2(2u_1+r_1-2b_1) \right\} \\ &+ 2(\Omega-\psi_1) \cot(\omega_I-\psi_I) \sin^2(\omega_I-\psi_I) = 0. \end{aligned} \tag{65}$$

Using the notations $t = \tan \omega_I$, $s = \tan \psi_I$ and substituting the first order solutions ψ_1, r_1, u_1, b_1 and A_I given in Eqs. (36 , 37, 38, 39 and 45) into Eq. (65), we solve for Ω as

$$\Omega = \frac{(1+st)M_I^2 tG}{2(M_I^2 t+s-t)} - 1 \qquad (66)$$

which, by virtue of β_1 given in Eq. (54), concludes the proof of Eq. (64). Hence the shock position, to be second order approximation, can be most easily determined. This simple and useful result is hardly surprising, it is a direct generalization of the well known fact in supersonic gasdynamics[14]. In fact it follows immediately from a theorem given by Lax* for a general class of differential equations satisfying conservation laws to which both Lunquist's equations and the equations governing gasdynamics (i.e. Euler's equations) belong. Our explicit calculations serve merely as a verification of this theorem.

III. Solutions for special cases

Taking limits of various parameters of the general solution, we obtain in this section, the first and second order solutions of simpler form for cases of (1) the gasdynamic limit (2) the

---

14. R. Courant and K.O. Friedrichs, Supersonic Flow and Shock Waves (Interscience Publishers Inc., New York, 1948).

* See footnote 11, page 6.

aligned field flow and (3) the switch on waves respectively. Several interesting features are illustrated. These solutions may be regarded as the zeroth order terms based on which a perturbation for slightly more complicated situations can be made. The agreement of these limiting solutions with various results derived directly from other simple formulations also serve to confirm the essential correctness of the general solution.

(1) The gasdynamic limit

To be precise, we mean the magnetic pressure being negligible compared with the gas pressure in the flow, or

$$\frac{P_m}{P} = \frac{B^2}{2\mu P} = \frac{\gamma b^2}{a^2} << 1 .$$

Keeping all physical quantities finite except $B^2 \rightarrow 0$, this is equivalent to

$$A_1^2 = U_I^2/b_I^2 \rightarrow \infty .$$

Now Eq. (45) reduces to

$$M_I^2 = 1+t^2/\dot{t}^2 . \qquad (67)$$

Since $a^2 << b^2$ in this limit, the cusped shape slow wave fronts become extremely small and are situated very near the origin, 0, while the outer fast wave front approximates very closely a circle. After taking the limit $A_I^2 \rightarrow \infty$, we obtain from Eqs. (36 to 40) the first order solutions

$$\psi_1 = \frac{(t-s)(1+st)}{t(1+s^2)} \tag{68}$$

$$r_1 = \frac{1+t^2}{t} \tag{69}$$

$$u_1 = -t \tag{70}$$

$$b_1 = \frac{(1+st)^2}{t(1+s^2)} \tag{71}$$

$$p_1 = \frac{\gamma(1+t^2)}{t} \quad . \tag{72}$$

From Eq. (56)

$$G = \gamma + 1 \tag{73}$$

simply. Thus the second order solutions in Eqs. (54, 55, 57 to 60) become

$$\beta_1 = \frac{(\gamma+1)(1+t^2)}{4} \tag{74}$$

$$\psi_2 = \frac{s[(\gamma+1)(1+s^2)(1+t^2)^3 - 4(t-s)(1+st)(st^2+2t-s)]}{4t^2(1+s^2)^2} \tag{75}$$

$$r_2 = \frac{(1+t^2)[(\gamma+1)t^4+3-\gamma]}{4t^2} \tag{76}$$

$$u_2 = -\frac{(\gamma+1)t^4 + 2(\gamma-1)t^2+(\gamma-1)}{4} \tag{77}$$

$$b_2 = \frac{(1+st)\left\{(\gamma+1)(st-1)(1+s^2)(1+t^2)^2+2(1+st)[2(1+s^2)+(t-s)^2]\right\}}{4t^2(1+s^2)^2} \tag{78}$$

$$p_2 = \frac{\gamma(1+t^2)[(\gamma+1)t^4 + 2(\gamma-1)t^2 + \gamma+1]}{4t^2} \quad . \tag{79}$$

Here s appears only in b_1, ψ_1, b_2 and ψ_2 as it should, so the magnetic field has no effect on the fluid motion. But the non-vanishing of b_1, ψ_1, b_2 and ψ_2 in general reveals that the magnetic field is affected by the flow. Hence the influence is only one way in this weak magnetohydrodynamic interaction limit. This is also evident from the basic differential equations*. As to Eqs. (67, 69, 70, 72, 74, 76, 77 and 79), they are the well-known gasdynamic result**.

(2) The aligned field flow

If the magnetic field has the same direction as that of velocity at any point in the flow, it must be so throughout the entire flow field. The problem is greatly simplified because of the disappearance of one of the basic physical parameters, namely ψ. Formally this corresponds to setting $\psi_1 \rightarrow 0$, or $s = \tan \psi_I \rightarrow 0$. Taking this limit of our general solution, we obtain from Eq. (45)

$$A_I^2 = \frac{(1+t^2)(M_I^2-1)}{M_I^2 t^2 - (1+t^2)} \tag{80a}$$

or

$$t^2 = \frac{M_I^2 + A_I^2 - 1}{(M_I^2-1)(A_I^2-1)} \tag{80b}$$

and the first order solutions from Eqs. (36 to 40)

---

* See footnote 8, page 5.

** See footnote 14, page 29.

$$\psi_1 = 1 \tag{81}$$

$$r_1 = \frac{M_I^2}{t(M_I^2-1)} \tag{82}$$

$$u_1 = \frac{-1}{t(M_I^2-1)} \tag{83}$$

$$b_1 = \frac{1}{t} \tag{84}$$

$$p_1 = \frac{\gamma M_I^2}{t(M_I^2-1)} \; . \tag{85}$$

For the second order solution, Eq. (56) becomes

$$G = \frac{3t^2M_I^4 + [\gamma-2(1+3t^2)]M_I^2 + 3(1+t^2)}{t^2M_I^2(M_I^2-1)} \; . \tag{86}$$

Then, we obtain from Eqs. (54, 55, 57 to 60)

$$\beta_1 = \frac{3t^2M_I^4 + [\gamma-2(1+3t^2)]M_I^2 + 3(1+t^2)}{4t^2(M_I^2-1)^2} \tag{87}$$

$$\psi_2 = 0 \tag{88}$$

$$r_2 = \frac{M_I^2\left\{(1-t^2)t^2M_I^4 + [2(1+t^4) + \gamma(t^2-1)]M_I^2-(3+t^4)\right\}}{4t^4(M_I^2-1)^3} \tag{89}$$

$$u_2 = \frac{(3+t^2-2\gamma)t^2M_I^4-[2(1+3t^2+t^4)-\gamma(1+t^2)]M_I^2+3+2t^2+t^4}{4t^4(M_I^2-1)^3} \tag{90}$$

$$b_2 = \frac{(1-t^2)t^2M_I^4 + [2(1+t^4)-\gamma(1+t^2)M_I^2 - (3+2t^2+t^4)}{4t^4(M_I^2-1)^3} \tag{91}$$

$$p_2 = \frac{\gamma M_I^2\left\{(2\gamma-1-t^2)t^2M_I^4+[2(1+t^2+t^4)-\gamma(1+t^2)]M_I^2-(3+t^4)\right\}}{4t^4(M_I^2-1)^3} . \tag{92}$$

Eqs. (81) and (88) indicate that the direction of magnetic field remains always along the flow direction indeed. In this degenerate case exact explicit solutions for both oblique shocks[15] and simple waves* of arbitrary strength are available. The latter corresponds to the invariants of the flow from which one may derive, as an alternative method, the above approximate solutions and even higher order terms of simple waves! Obviously the aligned field solution always contains the gas-dynamic solution as a limit. Putting $M_I^2 = \frac{1+t^2}{t^2}$ (i.e. Eq. (67) or Eq. (80b) at $A_i \rightarrow \infty$) in the above approximate solutions, we obtain readily the results given in section III(1) with s, of course, being set to zero in the expressions of b_1, ψ_1, b_2 and ψ_2.

(3) The switch-on wave

If the initial characteristic surface determined by upstream physical state is so oriented that along it the tangential magnetic field is absent whereas after the passage of the wave the tangential magnetic field along the new characteristic front has a finite magnitude; we speak for this situation the switch-on wave. In other words, the upstream state

15. J. Bazer and W.B. Ericson, "Oblique shock waves in a steady two dimensional hydromagnetic flow" in Proc. of Symp. on Electromagnetics and Fluid Dynamics of Gaseous Plasmas (Polytech. Inst. of Brooklyn, Brooklyn, N.Y., 1962).

* See footnotes 8 and 10, page 5.

has $$t = -\frac{1}{s} \tag{93}$$

as one of the solutions of Eq. (41) which corresponds to just one of the possible waves. It is meaningful, within the scope of the present analysis, to investigate only the switch-on simple waves for which the initial characteristics indeed describes the front of the first disturbance. Because the precise position of a shock is not known in priori and can only be determined here approximately by successive higher order solutions, the switch-on oblique magnetohydrodynamic shock can be adequately analyzed only from the exact jump relations across a discontinuous shock surface. It will be shown in next section that the switch-on waves actually correspond to one of the singular cases of the general solution. However, apart from the special case of switch-on waves at $M_I = A_I$ which results in an essential singularity, in all other cases they correspond only to a removable singularity as shown below.

Using Eq. (93) to eliminate t from Eq. (41) yields

$$[M_I^2 - (1+s^2)][A_I^2 - (1+s^2)] = 0 \ . \tag{94}$$

We assume $M_I^2 \neq A_I^2$ in our discussions here, so there are two different types of switch-on waves each corresponding to the vanishing of one of the brackets in Eq. (94)

(i) $$M_I^2 = 1+s^2 = \frac{1+t^2}{t^2} \ .$$

It is evident from Fig. 6 that the terminal of $-\vec{U}$ must lie on

(a) $\ell\ell$ or $\ell'\ell'$ when $a_I > b_I$ (or $M_I < A_I$), this corresponds to the fast wave mode or

(b) mm or m'm' when $a_I < b_I$ (or $M_I > A_I$), this corresponds to the slow wave mode.

Except for $A_I \neq M_I$, A_I can be completely arbitrary otherwise. In this singular case it is more convenient to consider the intermediate calculations and find ψ_1 first. Setting $t = -\frac{1}{s}$, we obtain from Eqs. (A-23 and -26), together with the definition of $\psi_1 = \phi_1/1+s^2$,

$$\left(1 - \frac{1+s^2}{A_I^2}\right)\psi_1 = 0 \tag{95}$$

$$\left(\frac{M_I^2}{A_I^2} - 1\right)\psi_1 = M_I^2 - (1+s^2) \tag{96}$$

respectively. Hence $\psi_1 = 0$. The other first order coefficients can be obtained from Eqs. (A-24,-25,-28 and -29) and similarly the second order solution. As an alternative method, we can also obtain the solution formally by first setting $M_I^2 = \frac{1+t^2}{t^2}$ and then taking the limit $1+st = 0$ of the coefficients given in Eqs. (36 to 40) and Eqs. (54 to 60). Then

(a) <u>The first order solution</u>

$$\psi_1 = 0, \quad r_1 = \frac{1+t^2}{t}, \quad u_1 = -t, \quad b_1 = 0, \quad p_1 = \frac{\gamma(1+t^2)}{t}$$

(b) The second order solution

$$G = \gamma+1 \quad , \qquad \beta_1 = \frac{\gamma+1}{4}(1+t^2)$$

$$\psi_2 = -\frac{(\gamma+1)(1+t^2)^2}{4t}$$

$$r_2 = \frac{(1+t^2)[(\gamma+1)t^4 + 3-\gamma]}{4t^2}$$

$$u_2 = -\frac{(\gamma+1)t^4 + 2(\gamma-1)t^2 + \gamma-1}{4}$$

$$b_2 = 0$$

$$p_2 = \frac{\gamma(1+t^2)[(\gamma+1)t^4 + 2(\gamma-1)t^2 + \gamma+1]}{4t^2}$$

Thus through this type of wave the magnetic field remains unchanged in its magnitude up to, while its direction does not change until, the second order of approximation. The coefficients of all other physical variables agree precisely with the gasdynamic solution*.

To see whether the tangential magnetic field along the new characteristic surface is indeed switched on, it suffices to examine if $F \neq 0$ where

$$F = (\psi_F - \psi_I) - (\dot{\omega}_F + \theta - \omega_1). \tag{97}$$

By the use of Eqs. (63 and 64), Eq. (97) becomes, to the first

* See footnote 14, page 29.

order of approximation,

$$F = (\psi_1 - 2\beta_1)\theta \ . \qquad (98)$$

Here

$$F = - \frac{\gamma+1}{2} (1+t^2)\theta \neq 0$$

in general. Hence the tangential magnetic field is switched on.

(ii) $$A_I^2 = 1+s^2 = \frac{1+t^2}{t^2} \ .$$

It is evident from Fig. 6 that the terminal of $-\vec{U}$ must lie on

(a) $\ell\ell$ or $\ell'\ell'$ when $a_I < b_I$ (or $M_I > A_I$), this corresponds to the slow wave mode or

(b) mm or m'm' when $a_I > b_I$ (or $M_I < A_I$), this corresponds to the fast wave mode.

Except for $M_I \neq A_I$, M_I can be completely arbitrary otherwise. In any of the above cases, the switch-on wave (fast or slow) always coincides with the Alfven wave mode. The latter does not enter in the restricted two-dimensional case considered here, but it plays an important role in the general two-dimensional case of orienting the magnetic field direction in the transverse plane (i.e. the plane perpendicular to the (x,y) plane) and renders an additional degeneracy. We have from Eqs. (95 and 96) for this case that

$$\psi_1 = A_I^2 = 1+s^2$$

the approximate solutions then follow immediately. Formally the solutions can also be obtained by setting $s = \frac{-1}{t}$ in the coefficients given in Eqs. (36 to 40) and Eqs. (54 to 60). They are

(a) <u>The first order solution</u>

$$\psi_1 = \frac{1+t^2}{t^2} = A_I^2 , \quad r_1 = b_1 = p_1 = 0 , \qquad u_1 = \frac{1}{t}$$

(b) <u>The second order solution</u>

$$G = 3, \qquad \beta_1 = 0$$

$$\psi_2 = \frac{(1+t^2)[M_I^2(t^2-1) - 2(1+t^2)]}{2t^3[M_I^2 t^2 - (1+t^2)]}$$

$$r_2 = \frac{M_I^2(1+t^2)^2}{2t^2[M_I^2 t^2 - (1+t^2)]}$$

$$u_2 = - \frac{M_I^2 + (1+t^2)(2+t^2)}{2t^2[M_I^2 t^2 - (1+t^2)]}$$

$$b_2 = \frac{(1+t^2)^2}{2t^4}$$

$$p_2 = \frac{\gamma M_I^2(1+t^2)^2}{2t^2[M_I^2 t^2 - (1+t^2)]} .$$

Therefore within the first order of approximation through this type of wave the density and pressure remain unchanged while the magnetic field, though retaining its magnitude, tilts

through a definite angle $A_I^2\theta$ with

$$A_I^2\,\theta > \theta \qquad (\text{or} < \theta) \quad \text{if} \quad \theta > 0 \qquad (\text{or} < 0)$$

for compression (or rarefaction) simple waves since $A_I^2 > 1$ always for nonaligned flows $(s \neq 0)$ in general. We obtain from Eq. (98) that

$$F = A_I^2\theta \neq 0$$

here. Hence the magnetic field is switched on along the new characteristics surface. It is evident from the above second order solution that an essential singularity arises at $A_I^2 = M_I^2 = \frac{1+t^2}{t^2}$.

IV. Singularity of the expansion

Each of the first and second order coefficients given in secion II has the common denominator

$$Q = M_I^2 t + s - t$$

always. At $Q \cong 0$, the coefficients become significantly large compared with order one unless the numerator has a comarably small value also. $Q = 0$ is in fact a singularity of the expansion near which a slight change in flow direction, θ , may result in a drastic change in state variables. This renders the series expansion of the assumed form to be no longer appropriate. We get, at $Q = 0$

$$t = \frac{s}{1-M_I^2} \quad . \tag{99}$$

If a singularity exists for a given state only one of the possible waves corresponding to Eq. (99) has the singular behavior. To determine all upstream states (characterized by M_I, A_I and s) which may give rise to such a singularity, we eliminate t from Eq. (41) by the use of Eq. (99) and obtain

$$(1+s^2-M_I^2)\left\{ s^2(M_I^2A_I^2-M_I^2-A_I^2) - (M_I^2-1)[M_I^2(M_I^2-1) - A_I^2]\right\} = 0. \tag{100}$$

Setting the first bracket of Eq. (100) to zero together with Eq. (99) yields

$$1 + st = 0 \tag{101}$$

This describes a switch on wave as discussed in the previous section. The resulting singularity is not essential and can be readily removed except for the special case of $M_I = A_I$, the latter will be discussed in more detail later in this section. We shall, in the following, mainly investigate the singularity corresponding to the vanishing of the second bracket in Eq. (100). It gives

$$s^2 = s_c^2(M_I^2,A_I^2) = \frac{(M_I^2-1)[M_I^2(M_I^2-1) - A_I^2]}{M_I^2A_I^2-M_I^2-A_I^2} \tag{102}$$

where we denote $s^2 = s_c^2 = s_c^2(M_I^2,A_I^2)$ being the trajectory of singularities with constant s_c^2 in (M_I^2,A_I^2) plane as shown in Fig. 7. It can be easily seen that the numerators of the

expansion coefficients under this condition are different from zero in general, hence the singularity is indeed essential. For any given physical state characterized by M_I, A_I and s, one may readily examine whether a singular wave does exist or not by comparing s^2 with s_c^2 determined by M_I^2 and A_I^2 as shown in Fig. 7.

Let us now define the parameter

$$k = \frac{M_I^2}{A_I^2} = \frac{b_I^2}{a_I^2} = \frac{B_I^2}{\gamma\mu P_I} = \frac{2}{\gamma}\frac{P_m}{P_I} \qquad (103)$$

as a measure of the magnetohydrodynamic interaction which is directly proportional to the ratio of magnetic and gas pressure. Eq. (102) becomes

$$k = \frac{(1+s^2)(M_I^2-1)}{s^2 + (M_I^2-1)^2} \quad . \qquad (104)$$

Since $k \geq 0$ always, we observe from Eq. (104) that the singularity exists only if $M_I \geq 1$. It is convenient to investigate the singularity in the hodograph plane where the characteristic locus is usually given. To this end, we write Eq. (104) as

$$k \cos^2\phi_I M_I^4 - (1+2k \cos^2\phi_I)M_I^2 + (1+k) = 0 \qquad (105)$$

for which the solution $M_I = M^* = M^*(\phi_I;k)$ is

$$M^*(\phi_I;k) = \left\{2(1+k)[1+2k\cos^2\phi_I \mp (1-k^2\sin^2 2\phi_I)^{\frac{1}{2}}]^{-1}\right\}^{\frac{1}{2}} . \qquad (106)$$

Evidently the singular surface is symmetric with respect to directions both along ($\phi_I = 0$ and π) and normal to ($\phi = \frac{\pi}{2}$ and $\frac{3\pi}{2}$) the magnetic field, it suffices to consider only the first quadrant with $\frac{\pi}{2} \geq \phi_I \geq 0$. We observe from Eq. (106) that $M^*(\phi_I;k)$ is real except when

$$1 < k^2 \sin^2 2\phi_I . \qquad (107)$$

The above inequality is always satisfied for $k < 1$ but not so for $k > 1$. By the use of the definition of k in Eq. (103), Eq. (102) takes the form

$$s_c^2 = \frac{k(M_I^2-1)[M_I^2-(1+k^{-1})]}{M_I^2-(1+k)} . \qquad (108)$$

Since $s_c^2 \geq 0$ and $M_I^2 \geq 1$ always, several qualitative features of the singular surface can be obtained from Eqs. (105 to 108).

(1) When $k < 1$, or the weak magnetohydrodynamic interaction case. M^* exists for all $\phi_I(\pi/2 \leq \phi_I \leq 0.)$ and

$$M^* \geq (1+k^{-1})^{\frac{1}{2}} \quad \text{or} \quad (1+k)^{\frac{1}{2}} \geq M^* \geq 1.$$

(2) When $k > 1$, or the strong magnetohydrodynamic interaction case. From Eq. (107), M^* exists only for ϕ_I satisfying

$$\tfrac{1}{2}\sin^{-1}(k^{-1}) \geq \phi_I \geq 0 \qquad \text{or} \qquad \frac{\pi}{2} \geq \phi_I \geq \frac{\pi}{2} - \tfrac{1}{2}\sin^{-1}(k^{-1})$$

and

$$M^* \geq (1+k)^{\frac{1}{2}} \qquad \text{or} \qquad (1+k^{-1})^{\frac{1}{2}} \geq M^* \geq 1 .$$

(3) When $k = 1$. This is a special case for which Eq. (105) can be factored into the simple form

$$(M_I^2 \cos^2\phi_I - 1)(M_I^2 - 2) = 0. \tag{109}$$

The singular surface in the entire hodograph plane now degenerates to two straight lines perpendicular to the magnetic field direction and a circle of radius $M_I = A_I = 2^{\frac{1}{2}}$. More generally, putting $A_I = M_I$ in Eq. (100) yields

$$(M_I^2 \cos^2\phi_I - 1)^2(M_I^2 - 2) = 0. \tag{110}$$

Therefore the vanishing of the first bracket of Eq. (110) pertains also to the singular case of a switch-on wave at which fast and slow wave coalesce, it is actually doubly degenerate[16].

To determine whether a singular wave is of fast or slow wave mode, we examine from the definitions of the fast and slow waves

---

16. In the general two dimensional case, this actually corresponds to a triple degeneracy at which fast, slow and transverse waves coalesce into a single wave front, the situation is highly singular indeed.

$$u_n \gtrless a \qquad \text{or} \qquad M_I^2 \sin^2\omega_I \gtrless 1$$

i.e.

$$(M_I^2-1)t^2 \gtrless 1. \tag{111}$$

By the use of Eq. (99) to eliminate t in Eq. (111), we get the criterion

$$M_I^2 \gtrless 1 + s^2 \qquad (\text{or} \quad M_I^2 \cos^2 \phi_I \lessgtr 1) \tag{112}$$

for fast and slow singular waves respectively.

In terms of the Cartesian coordinates in hodograph plane, $X = M_I \cos \phi_I$, $Y = M_I \sin \phi_I$, several additional interesting conclusions about the singular surface can be easily drawn. First of all, Eq. (112) becomes simply

$$X^2 \lessgtr 1 \tag{113}$$

for fast and slow singular waves respectively. Now Eq. (105) may be written as

$$Y^2 = \frac{(X^2-1)[X^2-(1+k^{-1})]}{k^{-1}-X^2} \quad . \tag{114}$$

Since $Y^2 \geq 0$, $1+k^{-1} \geq 1$ and $1 + k^{-1} > k^{-1}$ always, then $X^2 \leq 1 + k^{-1}$ always. Furthermore,

(1) When $k < 1$, the slow singular waves exist at

$$(1+k^{-1})^{\frac{1}{2}} \geq |X| > k^{-\frac{1}{2}} > 1$$

with $|X| = k^{-\frac{1}{2}}$ being the asymptotes for $|Y| \longrightarrow \infty$ and the fast singular waves exist at

$$1 \geq |X| \geq 0.$$

(2) When $k > 1$, the slow singular waves exist at

$$(1+k^{-1})^{\frac{1}{2}} \geq |X| \geq 1$$

and the fast singular waves exist at

$$1 > k^{-\frac{1}{2}} > |X| \geq 0$$

with $|X| = k^{-\frac{1}{2}}$ being the asymptotes for $|Y| \longrightarrow \infty$.

(3) When $k = 1$, Eq. (110) becomes

$$(X^2-1)^2(X^2+Y^2-2) = 0 \ . \tag{115}$$

Fast and slow waves coalesce at $|X| = 1$ now. On the circle $X^2 + Y^2 = 2$ fast waves exist at $|X| < 1$ and slow waves exist at $|X| > 1$.

Singular surfaces at different values of k are illustrated in Fig. 8. We see that the one at $k = 1$ is distinguished in the sense that it separates two families of singular surfaces having entirely different contours.

The singular surfaces for fast waves always lie outside or on the outer contour of the characteristic locus by the necessary condition of the existence of a fast wave. On the

other hand, the singular surfaces for slow waves may intersect the characteristic locus though $|M_I \cos \phi_I| > 1$ must always be satisfied.

It is parcitularly interesting to compare the singular surface with the characteristic locus at $k < 1$. Since $k^{\frac{1}{2}} < 1$ the slow singular surface is always outside of the characteristic locus. Let us now consider the fast singular wave surface, it generally intersects the fast wave branch (outer contour) of characteristic locus at two points, namely $(M_I, \phi_I) = (1, 0)$ and $((1+k)^{\frac{1}{2}}, \frac{\pi}{2})$, in the first quadrant. For the intermediate angles, $\frac{\pi}{2} > \phi_I > 0$, numerical calculations of the exact positions of the singular surface and characteristic locus[2] have been performed for various values of k, the result indicates that the fast wave branches of these two surfaces are extremely close at most values of k (see Fig. 9). The difference between them becomes substantial only when k is close to 1. In the special case of $k \rightarrow 0$ which corresponds to the gasdynamic limit, we have

$$M_I = 1 \tag{116}$$

describing fast wave branches of both singular surface and characteristic locus. In view of the general behaviors of these two surfaces, the exact coincidence here is rather surprising. Eq. (116) describes the familiar sonic circle in gasdynamics that marks the boundary separating flow fields of hyperbolic and elliptic types. It is well known that near

this sonic circle at $k = 0$, i.e. at $|M_I - 1| << 1$, the flow belongs to the transonic regime in which a transition of two different types of flow pattern takes place and a special scheme other than the linear perturbation is necessary.

At $k = 1$, the equation for the characteristic locus* has the simple form

$$\sin^2\phi_I(M_I^2\cos^2\phi_I - 1)^2[(M_I^2 - 2)(4M_I^2 + 1)^2 + 27M_I^2\cos^2\phi_I] = 0. \tag{117}$$

It can be readily compared with the equation for the singular surface at $k = 1$ in Eq. (110).

However, the behaviors of the singular surface and the characteristic locus are completely different at $k > 1$ (see Fig. 9).

Although the physical implication of its occurence is obscure, the singularity is directly associated with the perturbation procedure and must also arise when one performs a straightforward perturbation of the system of Lunquist's differential equations about an uniform state. To obtain an uniformly valid approximate solution for a flow involving a singular wave, an adequately chosen distorted physical coordinates related directly to the basic physical, as well as geometrical parameters of the problem is needed. This would yield a set of new approximate differential equations

---

* See footnote 2, page 4.

(not necessarily linear) together with their appropriate boundary conditions. A detailed investigation of this singular situation and the method of resolving it poses undoubtly an interesting problem for future research.

V. Concluding Remarks

It is familiar in supersonic gasdynamics that the departures of the approximate solutions for moderately strong oblique shocks and simple waves from their respective exact solutions are substantially reduced by inclusion of the second order terms; the approximation remains good even up to appreciably large θ [17]. In view of the same mathematical structure shared by Lunquist's and Euler's equations, we are led to expect that the present approximate solution would be equally successful to describe moderately strong magnetohydrodynamic oblique shocks and simple waves. A comparison in this regard cannot be made until an exact explicit solution of the magnetohydrodynamic oblique shocks is worked out while an exact explicit solution of the general magnetohydrodynamic simple waves is not likely possible. On the other hand, direct comparison with experimental result is suggestive. Among the simplest and the most appropriate ones to perform is a flow past nonconducting wedge of extended length at various angles of attack[18] where only shocks and rarefaction simple waves enter

---

17. E.V. Laitone, J. Aero. Sci. 14, 25 (1947).
18. Cf. C.K. Chu and Y.M. Lynn, AIAA J. 1, 1062 (1963) where a linear analysis is given. An extension of it to take into account the second order terms presented here is straightforward.

as basic constituents of the flow field.

Acknowledgements

The author is indebted to Professors J. Bazer and H. Grad for their encouragement.

This work is supported by the Courant Institute of Mathematical Sciences under Contract AT(30-1)-1480 with the U.S. Atomic Energy Commission.

Appendix

The procedures of obtaining the approximate solutions are straightforward, though rather tedious. Several essential steps taken to facilitate the calculations are supplied in this appedix. For some of the singular cases, e.g. the switch-on waves, the solution can be obtained more directly and unambiguously from the intermediate formulae. It is convenient, especially in the derivation of the complicated second order solution, to use the tangent of angles rather than the angles themselves. We have then, instead of the cumbersome trigonometric equations, only algebraic equations to be calculated. Let us define

$$\varepsilon = \tan\theta \quad , \qquad \tau = \tan\beta$$

$$\phi = \tan\psi_F \quad , \qquad s = \tan\psi_I \; .$$

Eqs. (23, 25 to 28) become

$$\overline{B} = \sqrt{\frac{1+\phi^2}{1+s^2}\,\frac{\tau-s}{\tau-\phi}} \tag{A-1}$$

$$\overline{U} = \frac{\sqrt{1+\varepsilon^2}}{1+\tau\varepsilon}\left[1+\frac{1}{A_I^2}\,\frac{(\phi-s)(1+\tau^2)(\tau-s)}{\tau(1+s^2)(\tau-\phi)}\right] \tag{A-2}$$

$$\overline{\rho} = \frac{\tau(1+\tau\varepsilon)}{\tau-\varepsilon}\left[1+\frac{1}{A_I^2}\,\frac{(\phi-s)(1+\tau^2)(\tau-s)}{\tau(1+s^2)(\tau-\phi)}\right]^{-1} \tag{A-3}$$

$$\frac{\overline{P}-1}{\gamma M_I^2} = \frac{\tau\varepsilon}{1+\tau\varepsilon}+\frac{1}{2A_I^2}\left\{1-\frac{\tau-s}{(1+s^2)(\tau-\phi)}\left[\frac{2(\phi-s)(\tau-\varepsilon)}{1+\tau\varepsilon}+\frac{(1+\phi^2)(\tau-s)}{\tau-\phi}\right]\right\} \tag{A-4}$$

$$\frac{s(1+\tau\varepsilon)}{\phi-\varepsilon} = \frac{\tau-s}{\tau-\phi}\left[1 + \frac{1}{A_I^2}\ \frac{(\phi-s)(1+\tau^2)(\tau-s)}{\tau(1+s^2)(\tau-\phi)}\right] . \tag{A-5}$$

The above equations are arranged in the form from which the gasdynamic limit $(A_I \rightarrow \infty)$ can be readily recovered. We may, with the aid of Eq. (A-5), simplify Eqs. (A-2 and -3) to

$$\overline{U} = \frac{s\sqrt{1+\varepsilon^2}\ (\tau-\phi)}{(\phi-\varepsilon)(\tau-s)} \tag{A-6}$$

$$\overline{\rho} = \frac{\tau(\tau-s)(\phi-\varepsilon)}{s(\tau-\phi)(\tau-\varepsilon)} \quad . \tag{A-7}$$

Eqs. (A-1, -4 to -7) toegether with Eq. (29) that

$$\overline{P} = \overline{\rho}^{\gamma} \tag{A-8}$$

constitute a closed system from which the first and second order approximate solutions can be easily derived. We first solve a primary system consisting of Eqs. (A-4, -5, -7, and -8) for the variables $\overline{\rho}$, $\overline{P}$, τ, ϕ and ε exclusively, $\overline{B}$ and $\overline{U}$ may be determined subsequently from Eqs. (A-1 and -6).

Making use of Eqs. (30 to 35) given in the text, we obtain the asymptotic expansions of physical variables in terms of $\varepsilon = \tan\theta = \theta + O(\theta^3)$ up to second order in ε as

$$\overline{U} = 1 + u_1\varepsilon + u_2\varepsilon^2 + O(\varepsilon^3) \tag{A-9}$$

$$\overline{B} = 1 + b_1\varepsilon + b_2\varepsilon^2 + O(\varepsilon^3) \tag{A-10}$$

$$\overline{\rho} = 1 + r_1\varepsilon + r_2\varepsilon^2 + O(\varepsilon^3) \tag{A-11}$$

$$\overline{P} = 1 + p_1\varepsilon + p_2\varepsilon^2 + O(\varepsilon^3) \tag{A-12}$$

$$\phi = s + \phi_1\varepsilon + \phi_2\varepsilon^2 + O(\varepsilon^3) \tag{A-13}$$

$$\tau = t + \tau_1\varepsilon + \tau_2\varepsilon^2 + O(\varepsilon^3) \tag{A-14}$$

where we have

$$\phi_1 = \psi_1 \sec^2\psi_I = \psi_1(1+s^2) \tag{A-15}$$

$$\phi_2 = (\psi_2 + \psi_1^2 \tan \psi_I)\sec^2\psi_I = (\psi_2+\psi_1^2 s)(1+s^2) \tag{A-16}$$

$$t = \tan \beta_o \tag{A-17}$$

$$\tau_1 = \beta_1 \sec^2\beta_o = \beta_1(1+t^2) \tag{A-18}$$

or conversely

$$\psi_1 = \frac{\phi_1}{1+s^2} \tag{A-19}$$

$$\psi_2 = \frac{1}{1+s^2}\left(\phi_2 - \frac{\phi_1^2 s}{1+s^2}\right) \tag{A-20}$$

$$\beta_1 = \frac{\tau_1}{1+t^2} . \tag{A-21}$$

(1) First order solution

Substituting the expansions in Eqs. (A-9 to -14) into Eqs. (A-1, -4 to -8) respectively and equating terms of order ε , we obtain from Eq. (A-4)

$$\frac{p_1}{\gamma M_I^2} = t - \frac{\phi_1(1+t^2)}{A_I^2(1+s^2)(t-s)} \qquad \text{(A-22)}$$

from Eq. (A-5)

$$1 + st = \left[\frac{t}{t-s} + \frac{s(1+t^2)}{A_I^2 t(1+s^2)}\right]\phi_1 \qquad \text{(A-23)}$$

from Eq. (A-7)

$$r_1 = \frac{t\phi_1}{s(t-s)} + \frac{s-t}{st} \qquad \text{(A-24)}$$

and from (A-8)

$$p_1 = \gamma r_1 \; . \qquad \text{(A-25)}$$

Elimination of r_1 and p_1 from Eqs. (A-22, -24 and -25) yields

$$M_I^2 t + \frac{t-s}{st} = \left[\frac{t}{s} + \frac{M_I^2(1+t^2)}{A_I^2(1+s^2)}\right]\frac{\phi_1}{t-s} \; . \qquad \text{(A-26)}$$

Further elimination of ϕ_1 from Eq. (A-23 and -26) yields the compatibility condition given in Eq. (41). To express the solution in terms of M_I, s and t, we use Eq. (45) to eliminate A_I^2 and obtain from either of Eqs. (A-23 and -26)

$$\phi_1 = \frac{(t-s)[M_I^2 t^2(1+s^2) - (t-s)^2]}{t^2(M_I^2 t+s-t)} \tag{A-27}$$

which, by the use of Eq. (A-19) gives ψ_1 in Eq. (36). r_1 and p_1 in Eqs. (37 and 40) are easily obtained from Eqs. (A-24 and -25) together with Eq. (A-27). For the other first order coefficients we obtain from Eq. (A-1)

$$b_1 = \frac{(1+st)\phi_1}{(1+s^2)(t-s)} \tag{A-28}$$

and from Eq. (A-6)

$$u_1 = \frac{1}{s}\left(1 - \frac{t\phi_1}{t-s}\right) \tag{A-29}$$

which, by the use of Eq. (A-27) results in b_1 and u_1 in Eqs. (39 and 38) respectively.

(2) Second order solution

Equating terms of order ε^2 after the substitution of Eqs. (A-9 to -14) into Eqs. (A-1, -4 to -8) respectively, we obtain from Eq. (A-4)

$$\begin{aligned}\frac{p_2}{\gamma M_I^2} = \frac{-(1+t^2)\phi_2}{A_I^2(1+s^2)(t-s)} + \left\{1 + \frac{\phi_1}{A_I^2}\left[\frac{1}{(t-s)^2} - \frac{1}{1+s^2}\right]\right\}\tau_1 \\ - \frac{3\phi_1^2(1+t^2)}{2A_I^2(1+s^2)(t-s)^2} + \frac{\phi_1(1+t^2)}{A_I^2(1+s^2)} - t^2\end{aligned} \tag{A-30}$$

from Eq. (A-5)

$$\left[\frac{t}{s(t-s)} + \frac{1+t^2}{A_I^2 t(1+s^2)}\right]\phi_2 + \left\{\phi_1\left[\frac{t^2-1}{A_I^2 t^2(1+s^2)} - \frac{1}{(t-s)^2}\right] -1\right\}\tau_1$$

$$+ \frac{\phi_1^2}{t-s}\left[\frac{t(2s-t)}{s^2(t-s)} + \frac{2(1+t^2)}{A_I^2 t(1+s^2)}\right] + \frac{(2+st)\phi_1}{s^2} - \frac{1+st}{s^2} = 0 \tag{A-31}$$

from Eq. (A-7)

$$r_2 = \frac{t\phi_2}{s(t-s)} - \left[\frac{1}{t^2} + \frac{\phi_1}{(t-s)^2}\right]\tau_1 + \frac{t\phi_1^2}{s(t-s)^2} + \frac{s-t}{st^2} \tag{A-32}$$

and from Eq. (A-8)

$$p_2 = \gamma(\frac{\gamma-1}{2} r_1^2 + r_2). \tag{A-33}$$

Eliminating successively p_2, r_2, A_I, r_1, and ϕ_1 by the use of Eqs. (A-33, -32, 45, 37 and A-27) respectively in Eqs. (A-30 and -31), we solve τ_1 and ϕ_2 explicitly in terms of M_I^2, s and t as

$$\tau_1 = \frac{M_I^2 t(1+t^2)(1+st)G}{4(M_I^2 t+s-t)} \tag{A-34}$$

and

$$\phi = \frac{s[M_I^2 t^2(1+s^2) - (t-s)^2]}{4t^4(M_I^2 t+s-t)^3}$$

$$\times \left\{M_I^2 t \left\{M_I^2 t^2[2(t-s)^2 + (1+t^2)(1+st)] - 2(1+t^2)(t-s)^2\right\} G - 4(t-s)[M_I^4 t^4 - M_I^2 t^2(t^2-s^2) - (t-s)^2]\right\} \tag{A-35}$$

with G being defined in Eq. (56). Then β_1 in Eq. (54) is obtained from Eqs. (A-21 and -34) and ψ_2 in Eq. (55) is obtained from Eqs. (A-20, -35 and -27). We also obtain r_2 in Eq. (57) by substituting Eqs. (A-16, -15 and -18) into Eq. (32). Elimination of r_1 from Eqs. (37 and A-33) yields p_2 in Eq. (60). For the other second order coefficients, we obtain from Eq. (A-1)

$$b_2 = \frac{(1+st)\phi_2}{(1+s^2)(t-s)} + \left\{ \frac{\phi_1}{1+s^2}\left[\frac{1+st}{(t-s)^2} + \frac{1}{2(1+s^2)}\right] - \frac{\tau_1}{(t-s)^2} \right\}\phi_1 \qquad \text{(A-36)}$$

and from Eq. (A-6)

$$u_2 = \frac{1}{2} - \frac{\phi_2 t}{s(t-s)} + \frac{(\phi_1 - 1)(\phi_1 t + s - t)}{s^2(t-s)} + \frac{\phi_1 \tau_1}{(t-s)^2} \qquad \text{(A-37)}$$

which, by the use of Eqs. (A-15,-16, and -8) give b_2 and u_2 in Eqs. (59 and 58) respectively.

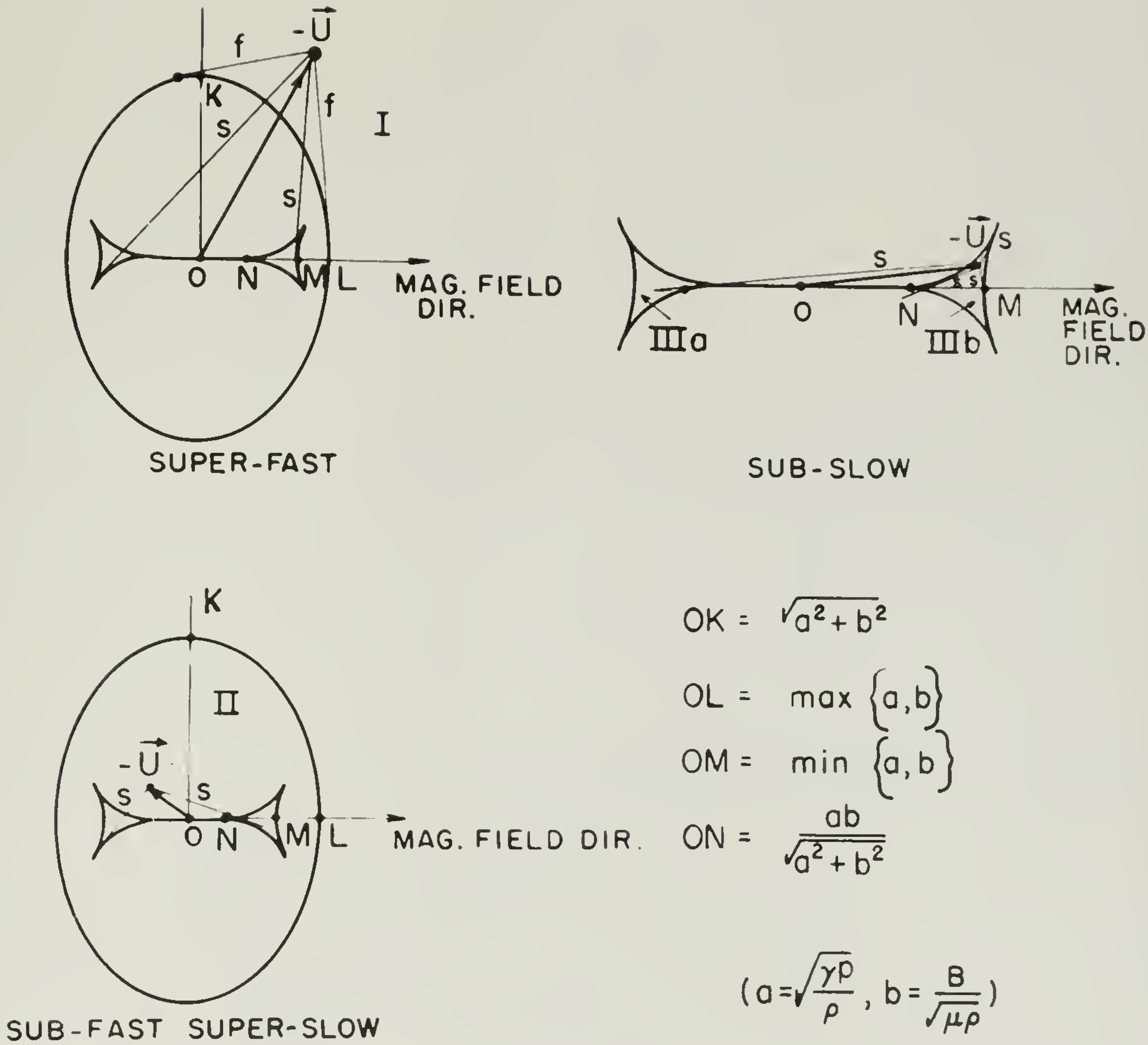

Fig. 1 Flow regimes

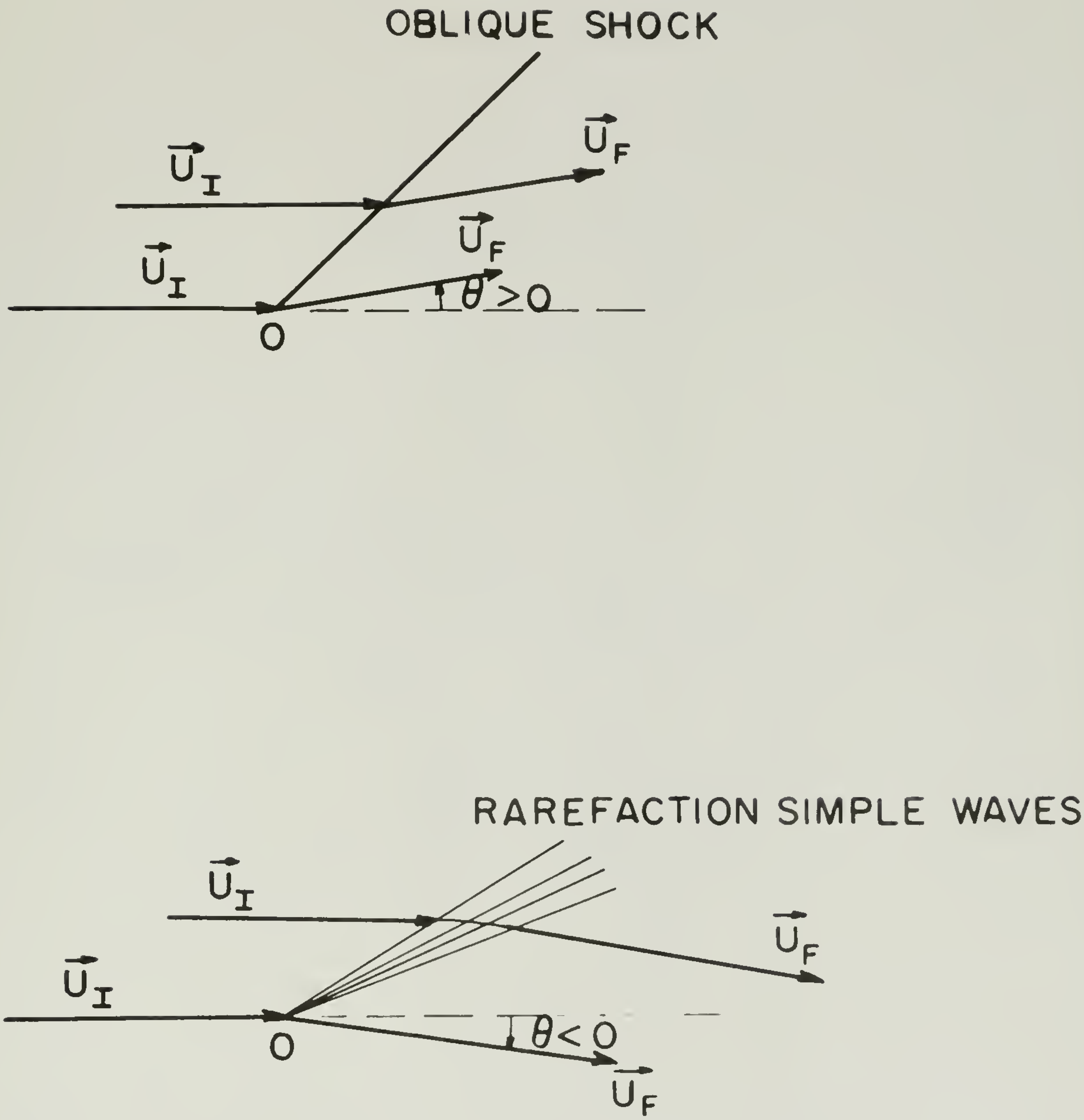

Fig. 2 Oblique shock and rarefaction simple waves

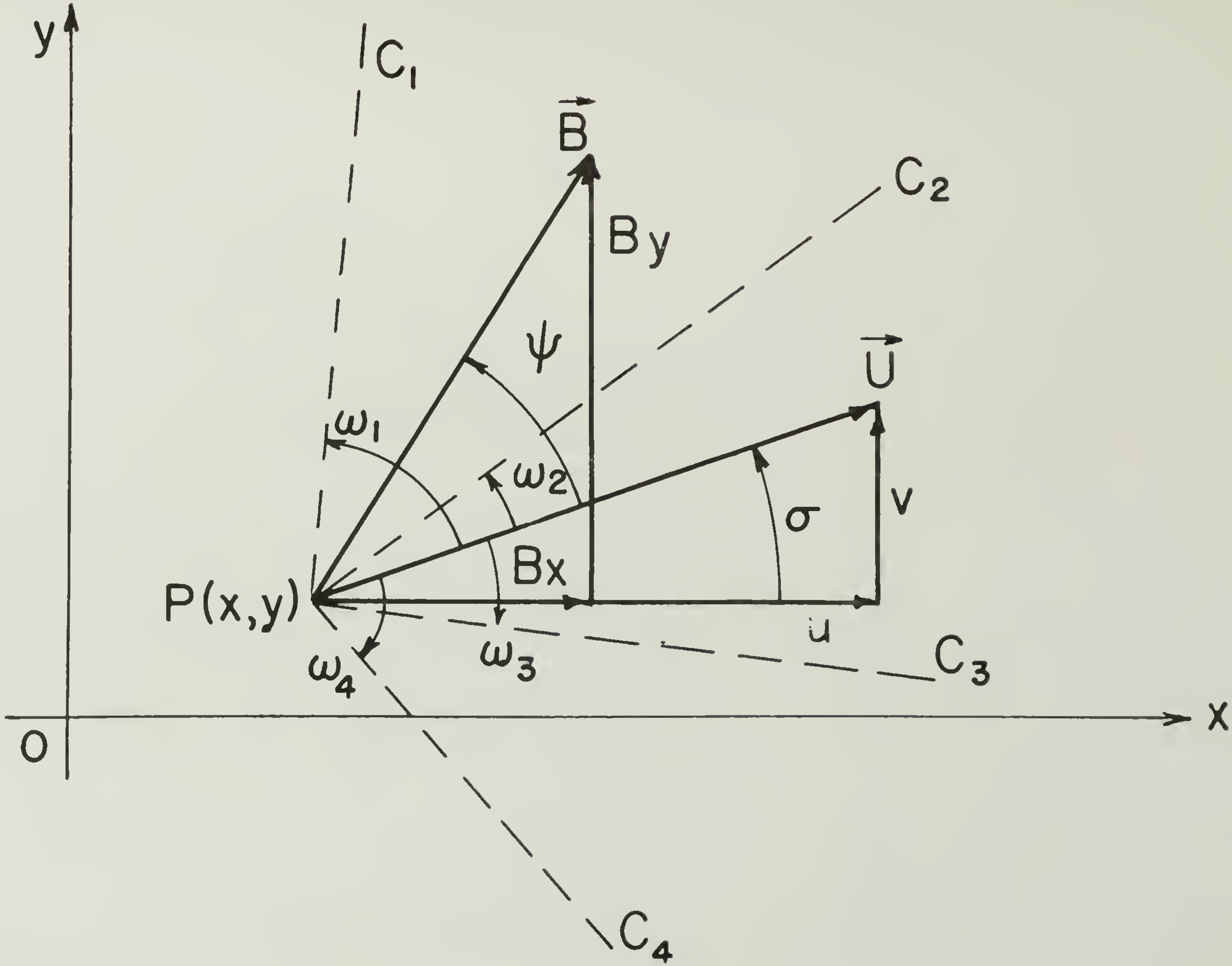

Fig. 3 Velocity, Magnetic Field and Characteristics

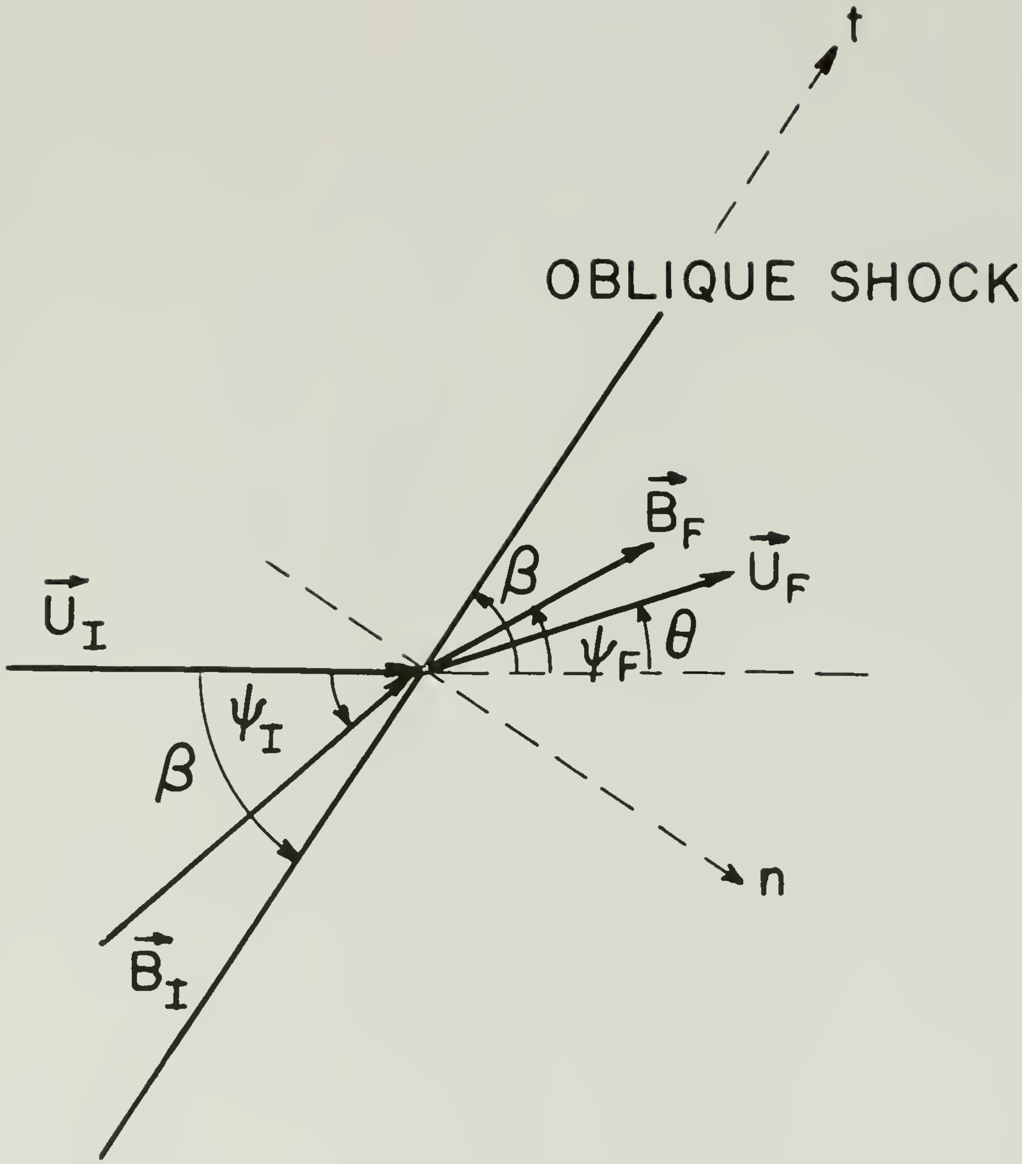

Fig. 4 Velocity and Magnetic Fiel Vectors across an oblique shock

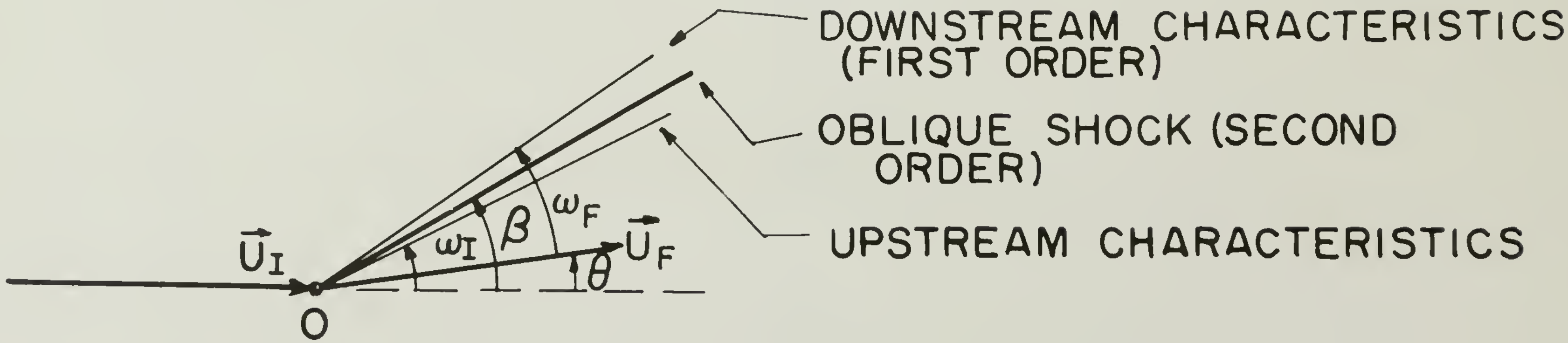

Fig. 5 Bisection of the angle between upstream characteristics and downstream characteristics (first order) by an oblique shock.

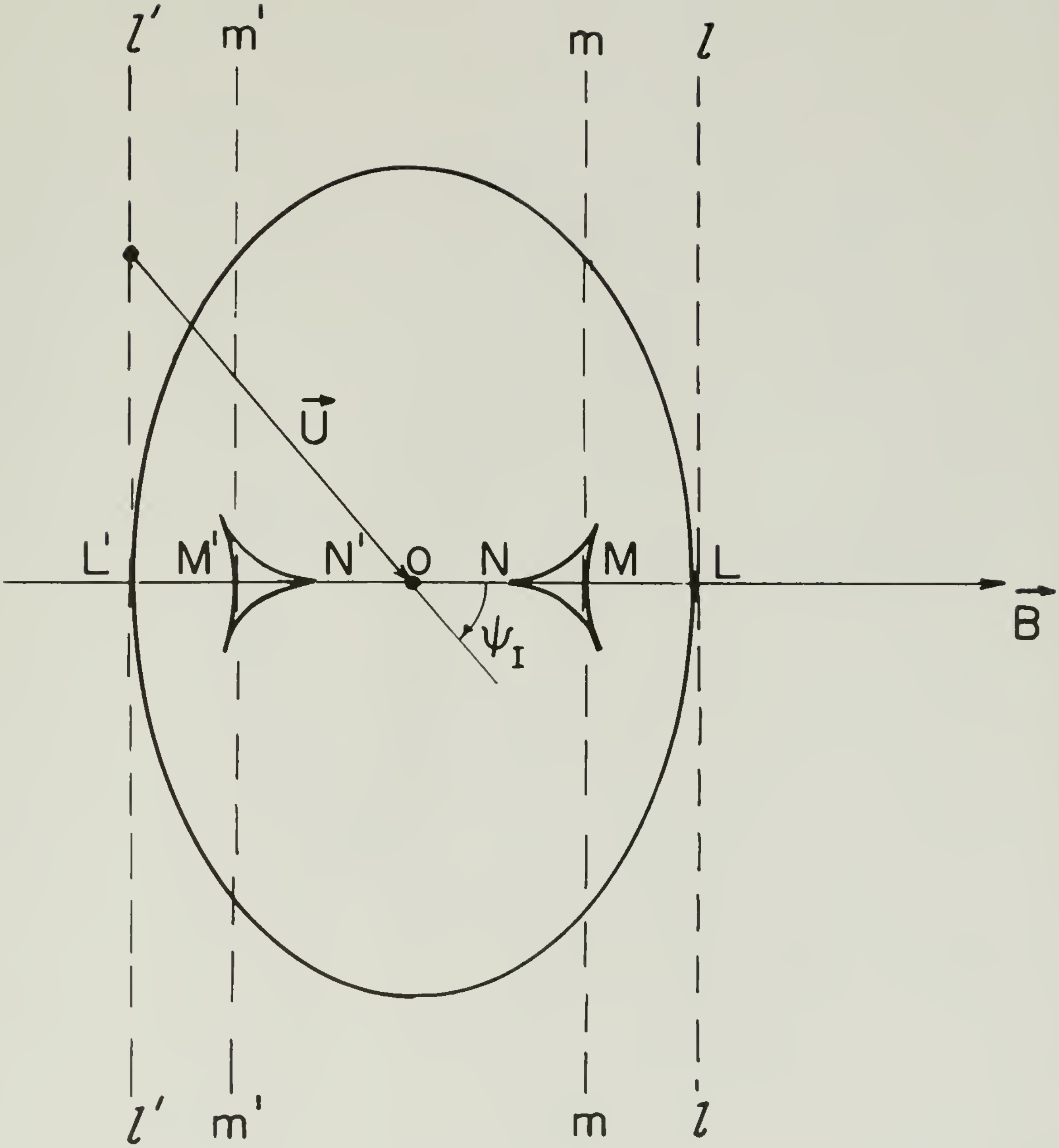

Fig. 6 Surfaces of switch-on waves

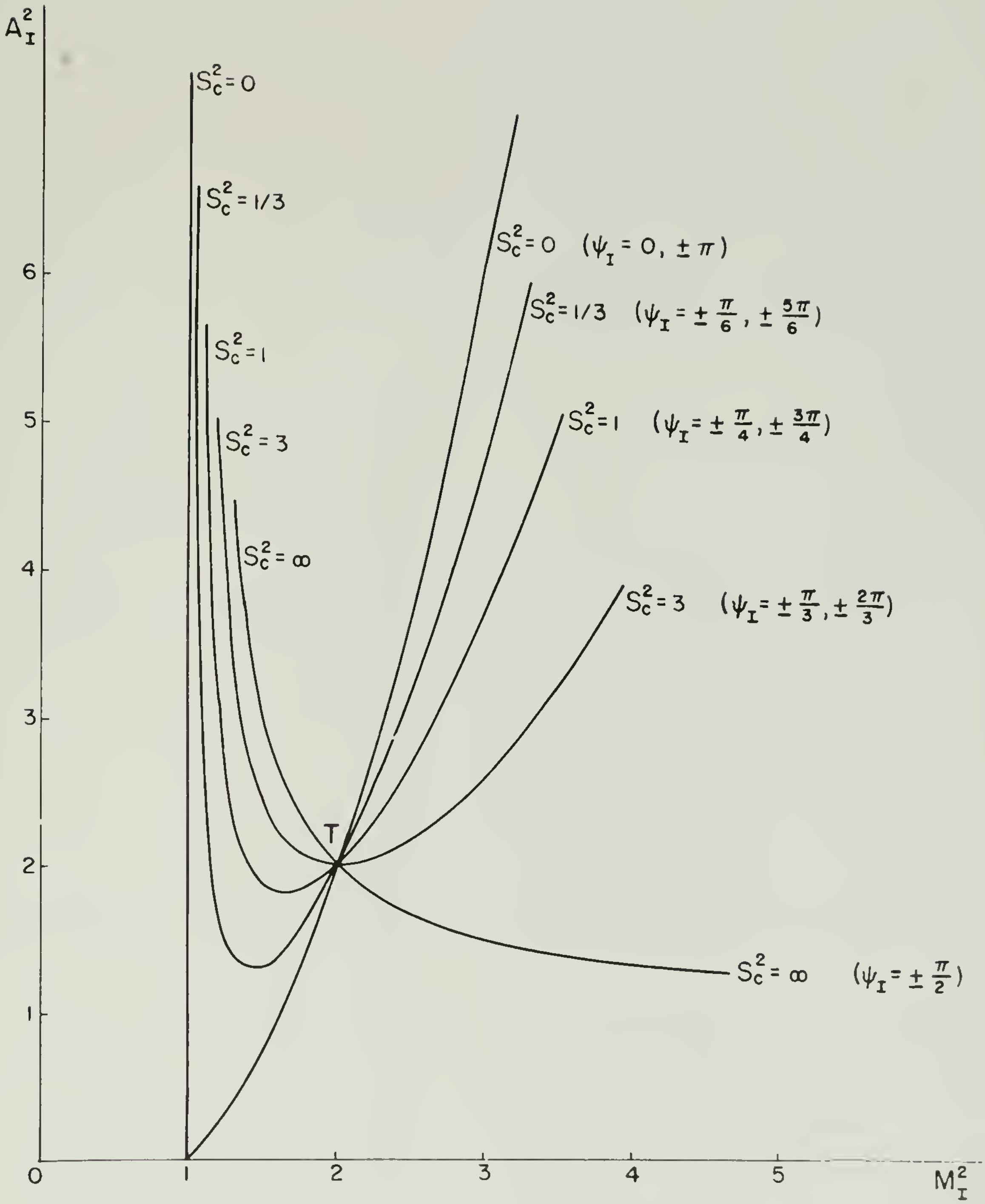

Fig. 7 The singular surface $S_c^2(M_I^2, A_I^2)$.

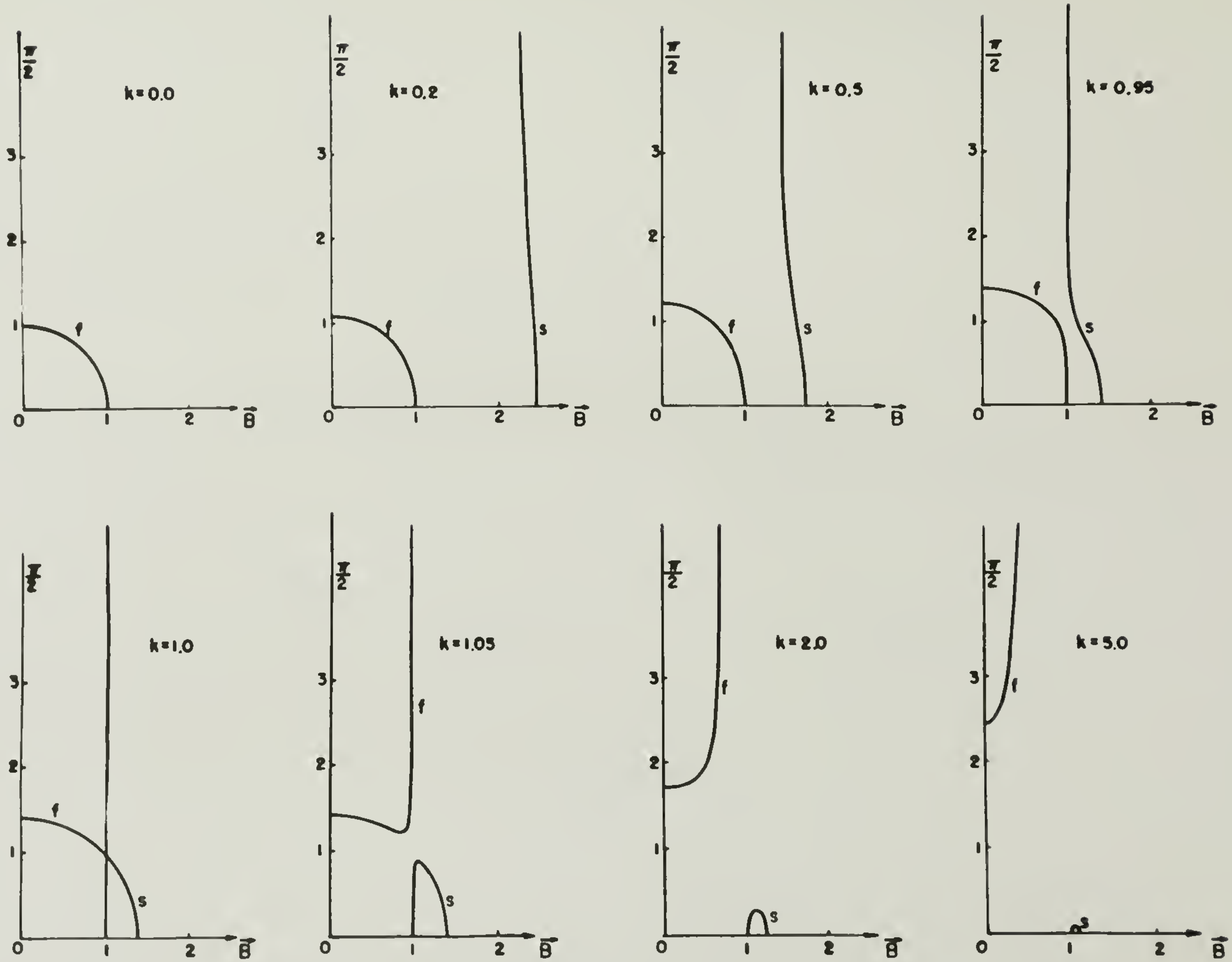

Fig. 8 Singular surfaces at various $k = M_I^2/A_I^2$ in hodograph plane.

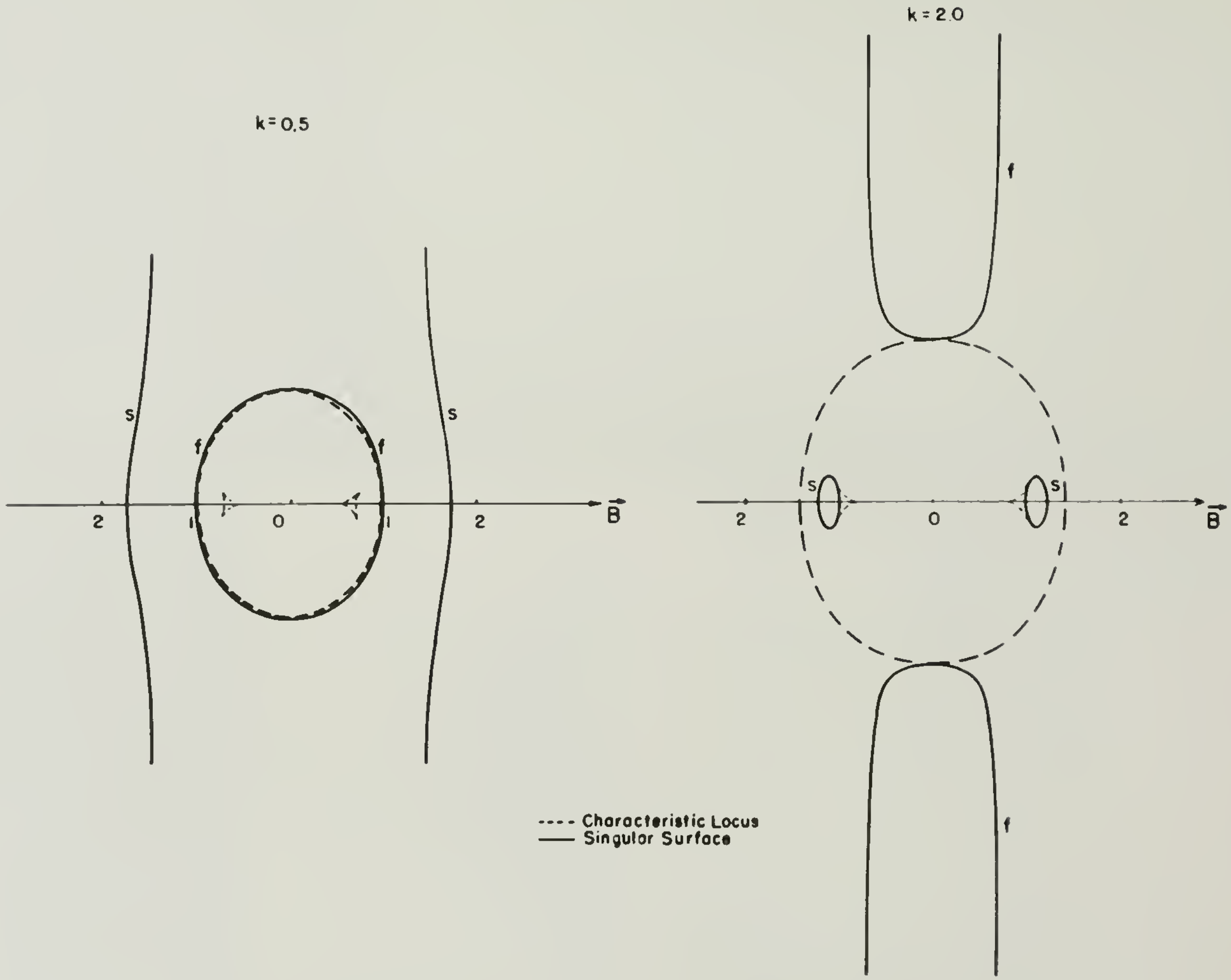

Fig. 9 Comparison of characteristic locus with singular surface at $k = 0.5$ and $k = 2.0$ (to scale except slight exaggeration at undistinguishable places).

4, 7

DATE DUE

GAYLORD PRINTED IN U.S.

NYU
NYO-
10,436
c.2

AUTHOR
Moderately strong oblique

TITLE
shocks...

DATE DUE	BORROWER'S NAME	ROOM NUMBER